Received: *October 8, 1999,* Accepted: *January 10, 2000*

Thermodynamics of spinning branes and their dual field theories

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Abstract: We present a general analysis of the thermodynamics of spinning black p-branes of string and M-theory. This is carried out both for the asymptotically-flat and near-horizon case, with emphasis on the latter. In particular, we use the conjectured correspondence between the near-horizon brane solutions and field theories with 16 supercharges in various dimensions to describe the thermodynamic behavior of these field theories in the presence of voltages under the R-symmetry. Boundaries of stability are computed for all spinning branes both in the grand canonical and canonical ensemble, and the effect of multiple angular momenta is considered. A recently proposed regularization of the field theory is used to compute the corresponding boundaries of stability at weak coupling. For the D2, D3, D4, M2 and M5-branes the critical values of Ω/T in the weak and strong coupling limit are remarkably close. Finally, we also show that for the spinning D3-brane the tree level R^4 correction supports the conjecture of a smooth interpolating function between the free energy at weak and strong coupling.

Keywords: Duality in Gauge Field Theories, p-branes, D-branes, Black Holes in String Theory.

*Work supported in part by TMR network ERBFMRXCT96-0045.

Contents

1. Introduction and conclusion

In the early 1970s, two important discoveries were made which have played a dominant role in theoretical physics ever since. The first discovery, by Bekenstein [1] and Hawking [2], was that four-dimensional black holes have thermodynamic properties due to Hawking radiation. Thus, by studying thermodynamics of black holes one probes the nature of quantum gravity. In the framework of string and M-theory, this discovery has been one of the main motivations to consider the thermodynamics of black p-branes [3, 4, 5]. The second discovery, by 't Hooft [6], was that non-abelian gauge theories simplify in the 't Hooft limit. In this limit the planar diagrams dominate and the theories thus become more tractable.

More recently, it has become clear that these two discoveries are in fact connected through the conjectured correspondence between the near-horizon limit of brane solutions in string/M-theory and certain quantum field theories in the large-N limit [7, 8]. As a consequence of this correspondence, studying the thermodynamic properties of black p-branes not only probes quantum gravity, but can in addition provide information about the thermodynamics of quantum field theories in the large-N limit.

In particular, for the non-dilatonic branes (D3,M2,M5) the near-horizon limit of the supergravity solutions has been conjectured [7] to be dual to a certain limit of the corresponding conformal field theories (see also refs. [9, 10] for an elaboration of the conjecture at the level of the partition function and correlation functions). In these AdS/CFT correspondences, the near-horizon background geometry is of the form $AdS_{p+2} \times S^{d-1}$ and the dual field theories are conformal. Moreover, for the more general dilatonic branes of type-II string theory preserving 16 supersymmetries, similar duality relations have also been obtained [8], which may be characterized more generally as Domain Wall/QFT correspondences [11, 12]. See ref. [13] for a comprehensive review and refs. [14, 15, 16] for some introductory lectures on the AdS/CFT correspondence and field theories in the large-N limit.

A common feature of these dualities between near-horizon backgrounds and field theories, is that the supergravity black p-brane solution exhibits an SO(d) isometry (where $d = D - p - 1$ is the dimension of the transverse space) which manifests itself as the R-symmetry of the dual field theory. As a consequence, by considering black p-brane solutions that rotate in the transverse space, we expect on the one hand to learn more about the field theory side, and on the other hand, be able to perform further non-trivial tests of the duality conjectures that include the dependence on this R-symmetry group. In particular, as will be reviewed below, the thermodynamics on the two sides provides a useful starting point for such a comparison.

The first construction of spinning branes solutions, rotating in the transverse space, can be found in refs. [17, 18, 19, 20] from which, in principle the most general black p-brane solution [21] can be derived by oxidization. Also, various spinning brane solutions [22]–[28] have recently been constructed and employed with the purpose to provide extra dimensionfull parameters in the decoupling of the unwanted KK modes in the context of obtaining QCD in various dimensions via the AdS/CFT correspondence [29]. Other examples of spinning brane solutions include the spinning NS5-brane [30] and rotating Kaluza Klein black holes [31]. Spinning branes have also been used [32] in the study of D-brane probes [33, 34, 35]. Many aspects of the case in which the rotation does not lie in the transverse space [36]–[40], generally referred to as the Kerr-AdS type, have also been considered in view of the AdS/CFT correspondence but will not be considered in this paper.

It is interesting in its own right to study the thermodynamic properties of black p-branes and rotating versions thereof, since this may teach us more about black brane physics. In particular, the near-horizon solution is thermodynamically much better behaved than the asymptotically-flat solution, which along with its relevance to a certain limit of the dual field theories, makes it very interesting to study the thermodynamics in this case. For the non-dilatonic branes the study of the thermodynamic stability has been initiated[1] in a number of recent papers [46]–[50]. The stability for the D3-brane with one non-zero angular momentum was addressed in [46] followed by an analysis both in the grand canonical and canonical ensemble for all non-dilatonic branes [47]. It was found that these two ensembles are not equivalent. An analysis of the critical behavior near these boundaries was also performed and shown to obey scaling laws of statistical physics. The case of multiple angular momenta was considered in ref. [50] for both ensembles.

Furthermore, in order to compare with the field theories, a regularization method [46, 50] has been proposed and used in order to compare the stability behavior obtained from the supergravity solution with that of the corresponding field theory in the weakly coupled limit. The angular momenta take values in the isometry group of the sphere, and hence map onto the R-charges in the dual field theory. Therefore, the angular velocities on the brane correspond in the field theory to voltages under the R-symmetry. In the presence of these voltages, a regularization is required, since for massless bosons with non-zero R-charge negative thermal occupation numbers occur. For the D3-brane case, it was found that the regulated field theory analysis predicts a similar upper bound on the angular momentum (or R-charge) density as obtained from the near-horizon brane solution. The critical exponents obtained from the supergravity solution are, however, not reproduced, though a mean field theory analysis has been suggested to cure this discrepancy. Finally, ref. [50] also presents evidence for localization of angular momentum on the brane outside the region of stability, and the occurrence of a first order phase transition.

The comparison of boundaries of stability in the two dual sides is one way to obtain evidence and predictions of the correspondence between near-horizon brane

[1]Other aspects of the thermodynamics in relation to the AdS/CFT correspondence and holography were studied e.g. in refs. [41]–[45].

solutions and field theories. Another route, that also uses thermodynamic quantities is consideration of the free energy which, in the non-rotating case, has been computed from the euclidean action by a suitable regularization method [51, 29]. For non-rotating D3, M2 and M5-branes it has been conjectured that there exists a smooth interpolating function connecting the two limits [52]. In particular for the D3-brane one finds that for the near-horizon $AdS_5 \times S^5$ limit the free energy differs by a factor 3/4 from the weakly coupled $N = 4$ SYM expression. Since the former limit corresponds to the strong 't Hooft coupling limit, it can be envisaged that higher derivative string corrections on the supergravity modify this result in such a way that minus the free energy increases towards the weak coupling limit. This conjecture was tested [52, 53] by computing the correction to the free energy arising from the tree level R^4 term in the type-IIB effective action, and shown to be in agreement. In this spirit, the study of such corrections is also interesting to perform in the presence of rotation.

In this paper, we will address various issues related to the developments described above, with emphasis on a general treatment for all black p-branes that are 1/2 BPS solutions of string and M-theory in the extremal and non-rotating limit. This includes the M2 and M5-branes of M-theory and the D and NS-branes of string theory. We will first write down the general asymptotically-flat solution of these spinning black p-brane in D dimensions. Since the transverse space is $d = D - p - 1$ dimensional, these spinning solutions are characterized by a set of angular momenta l_i, $i = 1, \ldots, n$ where $n = \mathrm{rank}(\mathrm{SO}(d))$, along with the non-extremality parameter r_0 and another parameter α related to the charge. Using standard methods of black hole thermodynamics we compute the relevant thermodynamic quantities of the general solution and show that the conventional Smarr formula is obeyed. (see section 2).

Our main interest, however, will be in the near-horizon limit of these spinning branes, which we will also compute in generality. The corresponding thermodynamics that results in this limit will also be obtained. In the near-horizon limit the charge and chemical potential become constant and are not thermodynamic parameters anymore, so that the thermodynamic quantities are given in terms of the $n + 1$ supergravity parameters (r_0, l_i). In particular, we derive and check a modified Smarr law for the near-horizon background which is due to a different scaling of the solution as compared to the asymptotically-flat case. One also finds a simple formula for the Gibbs free energy for any near-horizon spinning black p-brane solution with d transverse dimensions

$$F = -\frac{V_p V(S^{d-1})}{16\pi G} \frac{d-4}{2} r_0^{d-2} \,. \tag{1.1}$$

In a low angular momentum expansion, we rewrite this expression in terms of the intensive thermodynamic quantities, the temperature T and the angular velocities Ω_i. For comparison, we then use the correspondence with field theory in the large-N limit (and appropriate limit of the 't Hooft coupling limit in the case of D-branes), to write this free energy in terms of the field theory variables. (see section 3).

We proceed with presenting a general analysis of the boundaries of stability in both the grand canonical ensemble (with thermodynamic variables (T, Ω_i)) and the canonical ensemble (with thermodynamic variables (T, J_i)) of the near-horizon spinning branes. While there is a one-to-one correspondence between the $n+1$ supergravity variables and the extensive quantities (S, J_i), the map to the intensive ones (T, Ω_i) or the mixed combination (T, J_i) involves a non-invertible function, a fact which is crucial to the stability analysis. We will show in particular that for general d, the two ensembles are not equivalent and that increasing the number of equal-valued angular momenta enlarges the stable region.[2] For one non-zero angular momentum we find, in the grand canonical ensemble, that the region of stability (for $d \geq 5$) is determined by the condition

$$J \leq \sqrt{\frac{d-2}{d-4}} \frac{S}{2\pi}, \tag{1.2}$$

so that there is an upper bound on the amount of angular momentum the brane can carry in order to be stable. Put another way, at a critical value of the angular momentum density (which equals the R-charge density in the dual field theory) a phase transition occurs. The supergravity description also determines an upper bound on the angular velocity,

$$\Omega \leq \frac{2\pi}{\sqrt{(d-2)(d-4)}} T, \tag{1.3}$$

which is saturated at the critical value of the angular momentum. As a byproduct of the analysis we obtain an exact expression of the Gibbs free energy in terms of (T, Ω) for all branes in the case of one non-zero angular momentum.[3] We will also comment on the nature of the instability and discuss the setup to be solved in order to determine whether there is some region of parameter space in which phase mixing is thermodynamically favored, so that the angular momentum localizes on the brane. Finally, we give a uniform treatment of the critical exponents for all spinning branes in both ensembles and show that all of these are $1/2$, a value which satisfies scaling laws in statistical physics. (see section 4).

An important question is to what extent do we observe the above stability phenomena in the large-N limit of the dual field theory, also at weak coupling. To this end, extending the method of ref. [46], we obtain in an ideal gas approximation the free energies of the field theories for the case of the M-branes and the D-branes of type-II string theory. We review and extend the interpolation conjecture, stating

[2] Except for the cases $d = 8, 9$, which have no stability boundary for one non-zero angular momentum.

[3] The case $d = 4$, which can be seen from (1.1) to be special since the free energy vanishes, will be treated separately. In this case, the temperature and angular velocity are not independent, so that the phase diagram is degenerate.

JHEP01(2000)008

that the free energy smoothly interpolates between the weak and strong coupling limit. The free energies in the weakly coupled regime enable us to corroborate these conjectures by computing the boundaries of stability in this regime. The corresponding critical values of the dimensionless quantity Ω/T for the D2, D3, D4, M2 and M5-branes are remarkably close in the weak and strong coupling limit. (see section 5).

Finally, we also test the interpolation conjecture by considering the free energy. We first establish that for all near-horizon spinning branes the on-shell euclidean action reproduces the thermodynamically obtained Gibbs free energy (1.1). For the spinning D3-brane we then calculate, in a weak angular momentum expansion, the correction to the free energy due to the tree-level R^4 term in the type-IIB effective action. This order $\lambda^{-3/2}$ correction is positive (in the range of validity) and hence supports the conjectured existence of a smooth interpolating function between the free energy in the weak and strong coupling limit. (see section 6).

A number of appendices are included: appendix A gives a general discussion of (non-rotating) black p-branes, including those that preserve a lower amount of supersymmetry. We also find the thermodynamic quantities and Smarr formula for both the asymptotically-flat and near-horizon solutions. Appendix B reviews spheroidal coordinates which are relevant for the explicit form of spinning brane backgrounds. Appendix C shows how the euclidean spinning brane solution can be obtained from the minkowskian solution. Appendix D discusses the change of variables from the supergravity variables to the intensive thermodynamic variables in a weak angular momentum expansion. Finally, appendix E gives various useful expressions for the polylogarithms which are used in section 5 to compute the free energies of the weakly coupled field theories in the presence of voltage under the R-symmetry.

2. General spinning p-branes

More than forty years after the discovery of the Schwarzschild black hole metric, Kerr presented in 1963 the first metric for a rotating black hole [54]. About twenty years later, this was generalized to neutral rotating black holes of arbitrary dimensions in [55]. In [17, 18, 19] these were further generalized to charged rotating black hole solutions of the low-energy effective action of toroidally compactified string theory. In [20] the first spinning brane solutions appeared and recently spinning brane solutions of type-II string theory and M-theory have been presented in [22, 23, 24, 30, 21].

In this section we consider the general spinning brane solutions of string theory and M-theory. The general solution is presented in section 2.1 and in section 2.2 we derive the thermodynamic quantities of the general spinning brane solutions.

JHEP01(2000)008

2.1 The spinning black p-brane solutions

In this section we present the general charged spinning black p-brane solution with a maximal number of angular momenta for branes of string theory and M-theory. These solutions have the property that they are 1/2 BPS states in the extremal and non-rotating limit. Thus, they include the D- and NS-branes of 10-dimensional string theory[4] and the M-branes of 11-dimensional M-theory, as well as the branes living in toroidal compactifications of these theories. The solutions can be derived by oxidizing spinning charged black hole solutions in a $D-p$ dimensional space-time [17, 18, 19, 20, 22, 23, 24, 30, 21].

We only write the solutions for electric branes, since the magnetic solutions can easily be obtained by the standard electromagnetic duality transformation. In our conventions the coordinate system is taken to be (t, y^i, x^a), where t is the time, y^i, $i = 1, \ldots, p$ the spatial world-volume coordinates and x^a the transverse coordinates. The space-time dimension is denoted by D, so that $d = D - p - 1$ is the dimension of the transverse space. The spinning brane solutions given below are solutions of the action

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{g} \Big(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2(p+2)!} e^{a\phi} F_{p+2}^2 \Big) , \tag{2.1}$$

where F_{p+2} is the $(p+2)$-form electric field strength. This action arises as part of the 10-dimensional string effective action in the Einstein frame or the 11-dimensional supergravity action, and toroidal compactifications of these actions.[5] The value of a is a characteristic number for each brane, and for branes that are 1/2 BPS in the extremal and non-rotating limit, one has the relation

$$2(D-2) = (p+1)(d-2) + \frac{1}{2} a^2 (D-2) . \tag{2.2}$$

Appendix A reviews[6] more general black brane solutions that do not fulfill this identity and preserve a smaller amount of supersymmetry (See also table 6 for the values of a for each of the branes that we consider).

As described in appendix B, the spinning solutions depend on a set of angular momentum parameters $l_1, l_2, \ldots, l_n$ where $n = [d/2]$ is the rank of SO(d). Two further parameters that characterize the solutions are the non-extremality parameter r_0 and a dimensionless parameter α, related to the charge. In particular, $r_0 = 0$ corresponds

[4]Note that the D-branes of type-I string theory and the NS-branes of heterotic string theory are included in this class of branes. When discussing the dual field theories in the near-horizon limit we restrict to type-II string theory and M-theory only.

[5]Toroidal compactifications of the supergravity introduce more scalars in addition to the dilaton, which can be ignored since these moduli do not affect the background solution and its resulting thermodynamics.

[6]This appendix also gives the general thermodynamic relations for all non-rotating black p-branes, including those which preserve less than half of the supersymmetries.

to the extremal p-brane solution, while $\alpha = 0$ corresponds to a neutral brane. The relation of these parameters to the thermodynamic quantities of the solution will be discussed in section 2.2. We restrict ourselves to the cases for which the transverse dimensions lie in the range $3 \leq d \leq 9$ so that the brane solutions are asymptotically flat.

The metric of a charged spinning p-brane solution of the action (2.1) then takes the form

$$\begin{aligned} ds^2 = H^{-\frac{d-2}{D-2}} \left(- f dt^2 + \sum_{i=1}^{p} (dy^i)^2 \right) + H^{\frac{p+1}{D-2}} \left(\bar{f}^{-1} K_d dr^2 + \Lambda_{\alpha\beta} d\eta^\alpha d\eta^\beta \right) + \\ + H^{-\frac{d-2}{D-2}} \frac{1}{K_d L_d} \frac{r_0^{d-2}}{r^{d-2}} \left(\sum_{i,j=1}^{n} l_i l_j \mu_i^2 \mu_j^2 d\phi_i d\phi_j - 2 \cosh\alpha \sum_{i=1}^{n} l_i \mu_i^2 dt d\phi_i \right) . \end{aligned} \quad (2.3)$$

The electric dilaton is

$$e^\phi = H^{a/2} \quad (2.4)$$

and the electric potential A_{p+1} (with field strength $F_{p+2} = dA_{p+1}$) is given by

$$A_{p+1} = (-1)^p \frac{1}{\sinh\alpha} (H^{-1} - 1) \left(\cosh\alpha \, dt - \sum_{i=1}^{n} l_i \mu_i^2 d\phi_i \right) \wedge dy^1 \wedge \cdots \wedge dy^p . \quad (2.5)$$

Here, we have used spheroidal coordinates for the flat transverse space metric

$$\sum_{a=1}^{d} (dx^a)^2 = K_d dr^2 + \Lambda_{\alpha\beta} d\eta^\alpha d\eta^\beta \quad (2.6)$$

the explicit form of which can be found in appendix B, which also gives the angular dependence of the quantities μ_i. Moreover, we have defined

$$L_d = \prod_{i=1}^{n} \left(1 + \frac{l_i^2}{r^2} \right) , \qquad H = 1 + \frac{1}{K_d L_d} \frac{r_0^{d-2} \sinh^2\alpha}{r^{d-2}} \quad (2.7a)$$

$$f = 1 - \frac{1}{K_d L_d} \frac{r_0^{d-2}}{r^{d-2}} , \qquad \bar{f} = 1 - \frac{1}{L_d} \frac{r_0^{d-2}}{r^{d-2}} \quad (2.7b)$$

and we note that the harmonic function H is such that the branes are asymptotically flat.

The physical situation that this solution describes is a charged black p-brane rotating in the angles $\phi_1, \phi_2, \ldots, \phi_n$ (see appendix B). The rotation is static, meaning that the points of the p-brane move with time, but that the total set of points of the brane in the embedding space does not change with time. Thus, the solution describes a spinning charged black p-brane.

2.2 Thermodynamics of spinning branes

We proceed with describing some general physical properties of the solution given in (2.3), (2.4) and (2.5), and the computation of its relevant thermodynamic quantities.

The horizon is at $r = r_H$ where r_H is the highest root of the equation $\bar{f}(r) = 0$, so that

$$L_d(r_H) r_H^{d-2} = r_0^{d-2} \,, \tag{2.8}$$

where L_d is defined in (2.7a). On the other hand, the solution of the equation $f(r) = 0$ with the maximal possible value of r describes the so-called ergosphere, which coincides with the horizon for special values of the angles θ and $\psi_1, \psi_2, \ldots, \psi_{d-n-2}$. It is useful to find a coordinate transformation to a system in which these two hypersurfaces coincide. To this end we write

$$\tilde{t} = t \,, \qquad \tilde{\phi}_i = \phi_i - \Omega_i t \,, \qquad i = 1 \ldots n \,, \tag{2.9}$$

with all other coordinates unchanged. Thus, we want to find $\{\Omega_i, i = 1 \ldots n\}$ so that

$$g_{\tilde{t}\tilde{t}}\big|_{r=r_H} = 0 \,. \tag{2.10}$$

In the transformed frame one has

$$g_{\tilde{t}\tilde{t}} = g_{tt} + \sum_{i,j=1}^{n} \Omega_i \Omega_j g_{\phi_i \phi_j} + 2 \sum_{i=1}^{n} \Omega_i g_{t\phi_i} \tag{2.11}$$

so that (2.10) can be written as

$$g_{tt}\big|_{r=r_H} + \sum_{i,j=1}^{n} \Omega_i \Omega_j g_{\phi_i \phi_j}\bigg|_{r=r_H} + 2 \sum_{i=1}^{n} \Omega_i g_{t\phi_i}\bigg|_{r=r_H} = 0 \,. \tag{2.12}$$

Since eq. (2.12) should hold for all angles, we can consider the special choice for which $\mu_i = 1$ and $\mu_{j \neq i} = 0$. Then (2.12) becomes

$$g_{tt}\big|_{r=r_H} + \Omega_i^2 g_{\phi_i \phi_j}\big|_{r=r_H} + 2\Omega_i g_{t\phi_i}\big|_{r=r_H} = 0 \,, \qquad i = 1, \ldots, n \tag{2.13}$$

which is satisfied by

$$\Omega_i = \frac{l_i}{(l_i^2 + r_H^2)\cosh\alpha} \,, \qquad i = 1, \ldots, n \tag{2.14}$$

The new coordinate system defined in (2.9) can be seen as comoving coordinates on the horizon, i.e. coordinates for which the points on the brane in the embedding space do not move with time.[7] From the definition (2.9) it then follows that Ω_i is the

[7] In ref. [56] it was argued that the comoving frame is the natural frame for studying thermodynamics of rotating black holes and that the statistical analysis of rotating black holes is simplified in this frame.

JHEP01(2000)008

angular velocity of a particle on the horizon with respect to the angle ϕ_i. Thus, Ω_i is the angular velocity of the black p-brane with respect to the angle ϕ_i. Moreover, in the new coordinate system the off-diagonal metric component

$$g_{\tilde{t}\tilde{\phi}_i} = g_{t\phi_i} + \sum_{j=1}^{n} \Omega_j g_{\phi_j\phi_i} \tag{2.15}$$

has the property that it vanishes at the horizon

$$g_{\tilde{t}\tilde{\phi}_i}\big|_{r=r_H} = 0\,. \tag{2.16}$$

For completeness we also mention that the new coordinate system has the Killing vector

$$V \equiv \frac{\partial}{\partial\tilde{t}} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \Omega_i \frac{\partial}{\partial\phi_i}\,, \tag{2.17}$$

with norm $V^2 = g_{\tilde{t}\tilde{t}}$. It then follows from (2.12) that at the horizon this is a null Killing vector.

The new frame (2.9) also enables us to compute the temperature. By construction, both the metric components $g_{\tilde{t}\tilde{t}}$ and g^{rr} are zero for $r = r_H$. As a consequence, the standard procedure of transforming to euclidean space can be employed to find the temperature of the p-brane: First we go to the euclidean signature by a Wick rotation $\tau = i\tilde{t}$, and reinterpret the path-integral partition function as a partition function for a statistical system in $D-1$ dimensions with the temperature $T = 1/\beta$, where β is the periodicity of τ. This periodicity is determined by avoiding a singularity in space time (see for example [51]). In the case at hand, the euclidean metric near the horizon can be written as

$$ds^2 = -\partial_r g_{\tilde{t}\tilde{t}}\big|_{r=r_H}(r-r_H)d\tau^2 + \frac{1}{\partial_r g^{rr}|_{r=r_H}(r-r_H)}dr^2 + \cdots = \rho^2 d\Theta^2 + d\rho^2 + \cdots \tag{2.18}$$

with

$$\rho = 2\sqrt{\frac{r-r_H}{\partial_r g_{rr}|_{r=r_H}}}\,, \qquad \Theta = \frac{1}{2}\sqrt{-\partial_r g_{\tilde{t}\tilde{t}}|_{r=r_H}\partial_r g^{rr}|_{r=r_H}}\,\tau\,. \tag{2.19}$$

To avoid a conical singularity we need to require that Θ is periodic with period 2π, which determines

$$\beta = \frac{1}{T} = 4\pi \frac{1}{\sqrt{-\partial_r g_{\tilde{t}\tilde{t}}\big|_{r=r_H}\,\partial_r g^{rr}\big|_{r=r_H}}}\,. \tag{2.20}$$

Using (2.16) and (2.17) the formula for T can also be written as

$$T = \frac{1}{4\pi}\lim_{r\to r_H}\sqrt{\frac{g^{\mu\nu}\partial_\mu V \partial_\nu V}{-V^2}}\,. \tag{2.21}$$

One can then proceed to calculate the temperature T from (2.20) using the particular choice of angles $\theta = \pi/2$, so that $\mu_1 = 1$ and $\mu_{i\neq 1} = 0$. With this choice, we also have $\partial_\theta \mu_i^2 = \partial_{\psi_j}\mu_i^2 = 0$ for all i and j. After a tedious calculation, one obtains

$$T = \frac{d-2-2\kappa}{4\pi r_H \cosh\alpha}\,, \tag{2.22}$$

where we have defined

$$\kappa = \sum_{i=1}^{n} \frac{l_i^2}{l_i^2 + r_H^2}\,. \tag{2.23}$$

In fact, it follows[8] from (2.8) that $d-2-2\kappa \geq 0$ and hence $T \geq 0$. For $r_0 > 0$ it is thus possible to have $T=0$ and this in turn defines a boundary on the region of possible values of $(l_1, \ldots, l_n)$ in units of r_0.

Besides Ω_i and T, the chemical potential

$$\mu = -A_{y^1 y^2 \cdots y^p \tilde{t}}\Big|_{r=r_H} = \tanh\alpha \tag{2.24}$$

is also determined by the solution at the horizon. The ADM mass M and the charge Q of a spinning black p-brane are the same as for the non-rotating black p-brane, since we can measure these physical quantities in the asymptotic region of the space-time. In the asymptotic region, one can check that the metric (2.3) does not contain the angular momenta l_i to leading order in $1/r$. To calculate the ADM mass M one can therefore use the prescription given in ref. [57]. The angular momenta J_i, $i = 1 \ldots n$, can be read from the asymptotic expansion of (2.3) using the formula [55]

$$g_{t\phi_i} = -\frac{8\pi G}{V_p V(S^{d-1})}\frac{\mu_i^2}{r^{d-2}} J_i + \mathcal{O}\Big(\frac{\mu_i^2}{r^d}\Big)\,, \tag{2.25}$$

where

$$V(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{2.26}$$

is the volume of the $d-1$ dimensional unit sphere. Finally, the entropy S can be calculated from the Bekenstein-Hawking formula

$$S = \frac{A_H}{4G}\,, \tag{2.27}$$

where A_H is the area of the outer horizon. Alternatively, S can be found using the integrated Smarr formula reviewed below which follows from the 1st law of black-hole thermodynamics.

[8] To see this, define the function $h(x) = x^{d-2}\prod_{i=1}^n (1 + (l_i/x)^2) - r_0^{d-2}$ and compute $h'(x) = [d-2-2\sum_{i=1}^n l_i^2/(l_i^2+x^2)]x^{d-3}\prod_{i=1}^n(1+(l_i/x)^2)$. Using the fact that h and h' are both positive for large x, it follows that r_H cannot be the highest root of $h(x)=0$ if $d-2-2\kappa < 0$.

Summarizing, we list the complete set of thermodynamic quantities for a general spinning black p-brane

$$M = \frac{V_p V(S^{d-1})}{16\pi G} r_0^{d-2}\Big(d-1+(d-2)\sinh^2\alpha\Big) \tag{2.28a}$$

$$T = \frac{d-2-2\kappa}{4\pi r_H \cosh\alpha}\,, \qquad S = \frac{V_p V(S^{d-1})}{4G} r_0^{d-2} r_H \cosh\alpha \tag{2.28b}$$

$$\mu = \tanh\alpha\,, \qquad Q = \frac{V_p V(S^{d-1})}{16\pi G} r_0^{d-2}(d-2)\sinh\alpha\cosh\alpha \tag{2.28c}$$

$$\Omega_i = \frac{l_i}{(l_i^2 + r_H^2)\cosh\alpha}\,, \qquad J_i = \frac{V_p V(S^{d-1})}{8\pi G} r_0^{d-2} l_i \cosh\alpha \tag{2.28d}$$

where κ is defined in (2.23), V_p is the worldvolume of the p-brane and $V(S^{d-1})$ is the volume of the (unit radius) transverse $(d-1)$-sphere given in (2.26).

In further detail, the internal energy of a spinning charged black p-brane is the mass M. The other extensive thermodynamic parameters are the entropy S, the charge Q and the angular momenta $\{J_i\}$, and the first law of thermodynamics is

$$dM = TdS + \mu dQ + \sum_{i=1}^{n} \Omega_i dJ_i\,, \qquad M = M(S, Q, \{J_i\})\,. \tag{2.29}$$

Under the canonical scaling

$$r_0 \to \lambda r_0\,, \qquad l_i \to \lambda l_i\,, \qquad \alpha \to \alpha \tag{2.30}$$

we have the transformations

$$M \to \lambda^{d-2} M\,, \qquad S \to \lambda^{d-1} S\,, \qquad Q \to \lambda^{d-2} Q\,, \qquad J_i \to \lambda^{d-1} J_i\,. \tag{2.31}$$

It then follows from Euler's theorem that

$$(d-2)M = (d-1)TS + (d-2)\mu Q + (d-1)\sum_{i=1}^{n} \Omega_i J_i\,, \tag{2.32}$$

which is known as the integrated Smarr formula [58]. One can also reverse the logic and derive this formula using Killing vectors [55], and then use the scaling (2.30) to find (2.29). As an important check we note that the quantities listed in (2.28) indeed satisfy (2.32).

As an aid to the reader and for use below, we also give here the explicit expressions for the relevant parameters entering the solution (2.3) and the corresponding thermodynamics (2.28) for the M-branes in $D = 11$ and the D-branes in $D = 10$. To this end, it is useful to define the parameter h via the relation

$$h^{d-2} = r_0^{d-2}\cosh\alpha\sinh\alpha\,. \tag{2.33}$$

JHEP01(2000)008

Then, for the branes of M-theory we have the relations

$$16\pi G = (2\pi)^8 l_p^9\,, \qquad h^6 = 2^5\pi^2 N l_p^6 \quad (\mathrm{M2})\,, \qquad h^3 = \pi N l_p^3 \quad (\mathrm{M5})\,, \tag{2.34}$$

where l_p is the 11-dimensional Planck length and N the number of coincident branes. In parallel, for the Dp-branes of type-II string theory we record

$$16\pi G = (2\pi)^7 g_s^2 l_s^8\,, \qquad h^{d-2} = \frac{(2\pi)^{d-2} N g_s l_s^{d-2}}{(d-2)V(S^{d-1})} \quad (\mathrm{D}p)\,, \tag{2.35}$$

where l_s is the string length and g_s the string coupling.

3. Near-horizon limit of general spinning branes

In this section we examine the near-horizon limit of the spinning p-brane solutions considered in section 2. This provides further insights into the thermodynamics of black branes. More importantly, this is relevant since according to the correspondence [7, 8] between near-horizon brane solutions and field theories, this gives information about the strongly coupled regime of these field theories in the presence of non-zero voltages under the R-symmetry.

In section 3.1 we take the near-horizon limit while in section 3.2 we find the relevant thermodynamic quantities of the solution. Finally, in section 3.3 we discuss the map between the supergravity solutions and the dual field theories, obtaining in particular the free energies of these field theories.

3.1 The near-horizon solution

To find the near-horizon solution, one has to take an appropriate limit of the solution that is specified as follows: We introduce a dimensionfull parameter ℓ and perform the rescaling

$$r = \frac{r_{\mathrm{old}}}{\ell^2}\,, \qquad r_0 = \frac{(r_0)_{\mathrm{old}}}{\ell^2}\,, \qquad l_i = \frac{(l_i)_{\mathrm{old}}}{\ell^2}\,, \qquad h^{d-2} = \frac{h_{\mathrm{old}}^{d-2}}{\ell^{2d-8}} \tag{3.1a}$$

$$ds^2 = \frac{(ds^2)_{\mathrm{old}}}{\ell^{4(d-2)/(D-2)}}\,, \qquad e^{\phi} = \ell^{2a} e^{\phi_{\mathrm{old}}}\,, \qquad A = \frac{A_{\mathrm{old}}}{\ell^4}\,, \qquad G = \frac{G_{\mathrm{old}}}{\ell^{2(d-2)}} \tag{3.1b}$$

where the new quantities on the left hand side are expressed in terms of the old quantities labelled with a subscript "old", and we recall that h_{old} is defined in (2.33). Note that the rescaling in (3.1b) leaves the action (2.1) invariant due to the relation (2.2). The near-horizon limit is defined as the limit $\ell \to 0$ keeping all the new quantities in (3.1a) fixed. In particular, (3.1a) implies that in this limit we have $\frac{1}{4}e^{2\alpha} \to \ell^{-4}(h/r_0)^{d-2}$.

Using (2.3)–(2.5) the corresponding near-horizon solution then becomes

$$ds^2 = H^{-\frac{d-2}{D-2}}\left(- f dt^2 + \sum_{i=1}^{p} (dy^i)^2 \right) + H^{\frac{p+1}{D-2}} \left(\bar{f}^{-1} K_d dr^2 + \Lambda_{\alpha\beta} d\eta^\alpha d\eta^\beta \right) -$$
$$-2H^{-\frac{d-2}{D-2}} \frac{1}{K_d L_d} \frac{h^{\frac{d-2}{2}} r_0^{\frac{d-2}{2}}}{r^{d-2}} \sum_{i=1}^{n} l_i \mu_i^2 dt d\phi_i \tag{3.2a}$$

$$e^\phi = H^{\frac{a}{2}} \tag{3.2b}$$

$$A_{p+1} = (-1)^p \left(H^{-1} dt + \frac{r_0^{\frac{d-2}{2}}}{h^{\frac{d-2}{2}}} \sum_{i=1}^{n} l_i \mu_i^2 d\phi_i \right) \wedge dy^1 \wedge \cdots \wedge dy^p \,, \tag{3.2c}$$

where the harmonic function is now

$$H = \frac{1}{K_d L_d} \frac{h^{d-2}}{r^{d-2}} \tag{3.3}$$

and the functions L_d, K_d, f, $\bar{f}$ are as defined before in (2.7), (2.6), since the scale factor drops out in these expressions.

3.2 Thermodynamics in the near-horizon limit

We now turn to the thermodynamics of the near-horizon spinning p-brane solution (3.2) obtained in the previous subsection. Using the rescaling (3.1a) in the expressions (2.28b) and (2.28d) for (T, S) and (Ω_i, J_i) one finds in the near-horizon limit $\ell \to 0$ the following quantities,

$$T = \frac{d - 2 - 2\kappa}{4\pi r_H} \frac{r_0^{\frac{d-2}{2}}}{h^{\frac{d-2}{2}}} \,, \qquad S = \frac{V_p V(S^{d-1})}{4G} r_0^{\frac{d-2}{2}} h^{\frac{d-2}{2}} r_H \tag{3.4a}$$

$$\Omega_i = \frac{l_i}{(l_i^2 + r_H^2)} \frac{r_0^{\frac{d-2}{2}}}{h^{\frac{d-2}{2}}} \,, \qquad J_i = \frac{V_p V(S^{d-1})}{8\pi G} r_0^{\frac{d-2}{2}} h^{\frac{d-2}{2}} l_i \,. \tag{3.4b}$$

From (2.28c) we see that the chemical potential $\mu = 1$ and that the charge Q is constant. Thus, in the near-horizon limit the chemical potential and the charge are not anymore thermodynamic parameters.

In appendix A we derive the internal energy of a black p-brane in the near-horizon limit by defining this energy to be the energy above extremality $E = M - Q$. Since M and Q are not affected by the rotation of the brane, it follows from (A.15) that

$$E = \frac{V_p V(S^{d-1})}{16\pi G} \frac{d}{2} r_0^{d-2} \,. \tag{3.5}$$

The first law of thermodynamics for a spinning p-brane in the near-horizon limit is

$$dE = T dS + \sum_{i=1}^{n} \Omega_i dJ_i \,, \qquad E = E(S, \{J_i\}) \tag{3.6}$$

Under the canonical rescaling

$$h \to h\,, \qquad r_0 \to \lambda r_0\,, \qquad l_i \to \lambda l_i \tag{3.7}$$

we have the transformation properties

$$E \to \lambda^{d-2} E\,, \qquad S \to \lambda^{d/2} S\,, \qquad J_i \to \lambda^{d/2} J_i \tag{3.8}$$

as follows from (3.5) and (3.4). The scalings (3.8) imply with Euler's theorem the integrated Smarr formula for the near-horizon solution

$$(d-2)E = \frac{d}{2} TS + \frac{d}{2} \sum_{i=1}^{n} \Omega_i J_i \tag{3.9}$$

Remark that this conservation law for the near-horizon solution is different from the Smarr formula (2.32) of the asymptotically-flat solution, due to the different scaling behavior. It is not difficult to obtain the energy function of the microcanonical ensemble in terms of the extensive variables using the horizon equation (2.8) and (3.4), yielding

$$E^{d/2} = \left(\frac{d}{2}\right)^{d/2} \left(\frac{V_p V(S^{d-1})}{16\pi G}\right)^{-(d-4)/2} h^{-(d-2)^2/2} \left(\frac{S}{4\pi}\right)^{d-2} \prod_i \left(1 + \left(\frac{2\pi J_i}{S}\right)^2\right). \tag{3.10}$$

For later use we also calculate the Gibbs free energy

$$F = E - TS - \sum_{i=1}^{n} \Omega_i J_i = -\frac{d-4}{d} E = -\frac{V_p V(S^{d-1})}{16\pi G} \frac{d-4}{2} r_0^{d-2}\,, \tag{3.11}$$

which satisfies the thermodynamic relation

$$dF = -SdT - \sum_{i=1}^{n} J_i d\Omega_i\,, \qquad F = F(T, \{\Omega_i\})\,. \tag{3.12}$$

A remark is in order here for the special case $d = 4$, which includes the D5 and NS5-brane in 10 dimensions, since in that case it follows from (3.11) that $F = 0$. From (3.12) we observe that since the partial derivatives of F with respect to T and $\{\Omega_i\}$ are nonzero, these variables cannot be independent. Thus the phase diagram in terms of these variables degenerates into a submanifold with at least one dimension less. This point will be further illustrated in section 4.2 where we discuss the phase diagram for one non-zero angular momentum. For the non-rotating case, one immediately deduces from (3.4a) that the temperature must be constant for $d = 4$.

Since the Gibbs free energy is properly given in terms of the intensive quantities T and $\{\Omega_i\}$ we need to write the expression (3.11) in terms of these variables.[9] The change of variables from the supergravity variables $(r_0, \{l_i\})$ to these thermodynamic variables is given in appendix D in a low angular momentum expansion

$$\frac{l_i}{r_0} \ll 1 \tag{3.13}$$

through order $\mathcal{O}(l_i^4)$. Using the result (D.7) we find that

$$\begin{aligned} F = & -\frac{V_p V(S^{d-1})}{16\pi G}\frac{d-4}{2}\tilde{T}^{(2d-4)/(d-4)} h^{(d-2)^2/(d-4)} \times \\ & \times \left[1 + \frac{2}{d-4}\sum_i \tilde{\omega}_i^2 - \frac{2(d-6)}{(d-2)(d-4)^2}\left(\sum_i \tilde{\omega}_i^2\right)^2 + \frac{1}{d-4}\sum_i \tilde{\omega}_i^4 + \cdots \right], \end{aligned} \tag{3.14}$$

for $d \neq 4$ where we have defined $\tilde{T} = 4\pi T/(d-2)$ and $\tilde{\omega}_i = \Omega_i/\tilde{T}$.

As will be explained in section 3.3, F is the free energy for the field theory living on the brane in the strongly coupled large-N limit. In section 5 we compare this expression with the corresponding expressions in the weakly coupled field theory. Moreover, in section 6.1 we show that the free energy (3.11) is reproduced by calculating the (regularized) euclidean action of the solution.

3.3 The dual field theories

In the remainder of this paper, we will restrict ourselves to the spinning brane solutions of type-II string theory and M-theory. For these p-branes we can map [7, 8] the near-horizon limit to a dual Quantum Field Theory (QFT) with 16 supercharges, namely the field theory that lives on the particular p-brane in the low-energy limit. As explained in refs. [7, 8], in the near-horizon limit the bulk dynamics decouples[10] from the field theory living on the p-brane, so that the supergravity solution in the near-horizon limit describes the strongly coupled large-N limit of the dual QFT.

The fact that the branes are spinning, introduces the new thermodynamic parameters Ω_i and J_i on the supergravity side which need to be mapped to the field theory side, where they are conjectured to correspond to voltage and charge for the field theory R-symmetry group. Thus, the validity of this correspondence requires the R-symmetry groups to be $\mathrm{SO}(d)$, with the charges J_i taking values in the Cartan subgroup $\mathrm{SO}(2)^n$ of $\mathrm{SO}(d)$ and their Legendre transforms corresponding to the voltages Ω_i. In section 5 we analyze the field theory in the weakly coupled regime using the R-charge quantum numbers of the massless degrees of freedom.

Indeed, for the p-branes with $p \leq 6$ it has been noted in [8, 11] that the dual QFTs have the correct R-symmetry groups. In particular, in ref. [8] it was noted that Dp-branes have an $\mathrm{ISO}(1,p) \times \mathrm{SO}(d)$ symmetry in the near-horizon limit, where

[9] In section 4.2 we obtain the exact expressions for one non-zero angular momentum.
[10] With the exception of the D6-brane, as discussed for example in [8].

SO(d) corresponds to the R-symmetry group of the dual field theory and ISO$(1,p)$ corresponds to the Poincaré symmetry of the dual field theory. Moreover, in the dual frame, as considered in ref. [11] (see also [12]) the near-horizon solutions under consideration can be written as $DW_{p+2} \times S^{d-1}$ with a linear dilaton field, where DW_{p+2} is the $p+2$ dimensional Domain-Wall. Also in this language, the isometry group SO(d) of S^{d-1} translates into the R-symmetry group of the dual field theory.

As explained in [8, 12] we can trust the supergravity description of the dual field theory, when the string coupling $g_s \ll 1$ and the curvatures of the geometry are small. This implies in all cases that the number of coincident p-branes $N \gg 1$. For the M2- and M5-brane this is the only requirement since there is no string coupling in 11-dimensional M-theory. For the Dp-branes in 10 dimensions one must further demand that [8]

$$1 \ll g_{\rm eff}^2 \ll N^{\frac{4}{7-p}}\,, \qquad g_{\rm eff}^2 = g_{\rm YM}^2 N r^{p-3}\,, \tag{3.15}$$

where $g_{\rm eff}^2$ is the effective coupling, $g_{\rm YM}^2$ the coupling of the Yang-Mills theory on the Dp-brane and r is the rescaled radial coordinate in the near-horizon limit (the distance to the D-brane probe) and the Higgs expectation value in the dual QFT.[11] Thus, the near-horizon limit describes the dual QFT in the large-N and strongly coupled limit. Note that the thermodynamic expressions are valid when r is replaced by r_H in eq. (3.15). For larger values of the effective coupling the D1 and D5-brane flow to the NS1 and NS5-brane respectively, while the self-dual D3-brane flows to itself [8]. In particular, the NS1-brane description is valid for $N^{2/3} \ll g_{\rm eff}^2 \ll N$ and the NS5-brane description for $N^2 \ll g_{\rm eff}^2$ (see also refs. [59, 30] for further details on the type-II NS5-branes).

In view of this correspondence, we can write the Gibbs free energy and other thermodynamic quantities in terms of field theory variables.[12] In particular we need to specify the relation between the parameter ℓ entering the near-horizon limit and the relevant length scale of the theory and compute the rescaled quantities in (3.1a). In the following N is the number of coincident branes and we have defined the quantity $\omega_i = \Omega_i/T$.

For the M2-brane we need the relations

$$\ell = l_p^{3/4}: \quad 16\pi G = (2\pi)^8\,, \qquad h^6 = 2^5\pi^2 N\,, \tag{3.16}$$

where l_p is the 11-dimensional Planck length and we have used (2.34). Using this in (3.14) gives the Gibbs free energy

$$F_{\rm M2} = -\frac{2^{7/2}\pi^2}{3^4} N^{3/2} V_2 T^3 \left[1 + \frac{9}{8\pi^2}\sum_{i=1}^4 \omega_i^2 - \frac{27}{128\pi^4}\left(\sum_{i=1}^4 \omega_i^2\right)^2 + \frac{81}{64\pi^4}\sum_{i=1}^4 \omega_i^4 + \cdots \right]. \tag{3.17}$$

[11] See e.g. [8][32]–[35] for discussions of D-brane probes, including thermal and spinning D-branes.

[12] See [12] for a more detailed explanation of the mapping between the near-horizon supergravity solutions and QFTs, including a description of the cases with $D < 10$.

JHEP01(2000)008

For the M5-brane we have

$$\ell = l_p^{3/2}: \quad 16\pi G = (2\pi)^8, \qquad h^3 = \pi N, \tag{3.18}$$

giving

$$F_{\rm M5} = -\frac{2^6\pi^3}{3^7}N^3V_5T^6\left[1+\frac{9}{8\pi^2}\sum_{i=1}^2\omega_i^2+\frac{27}{128\pi^4}\left(\sum_{i=1}^2\omega_i^2\right)^2+\frac{81}{256\pi^4}\sum_{i=1}^2\omega_i^4+\dots\right]. \tag{3.19}$$

For the Dp-brane of type-II string theory we have from (2.35)

$$\ell = l_s: \quad h^{d-2} = \frac{(2\pi)^{2d-9}}{(d-2)V(S^{d-1})}\lambda, \qquad \frac{V(S^{d-1})h^{2(d-2)}}{16\pi G} = \frac{(2\pi)^{2d-11}}{(d-2)^2V(S^{d-1})}N^2, \tag{3.20}$$

where l_s, g_s are the string length and coupling and $\lambda = g_{\rm YM}^2 N$ is the 't Hooft coupling with the Yang-Mills coupling given by $g_{\rm YM}^2 = (2\pi)^{p-2}g_s l_s^{p-3}$. Using these relations in (3.14) we obtain (for $p \neq 5$)

$$F_{{\rm D}p} = -c_pV_pN^2\lambda^{-\frac{p-3}{p-5}}T^{\frac{2(7-p)}{5-p}}\left[1+\frac{S_p^1}{\pi^2}\sum_i\omega_i^2+\frac{S_p^2}{\pi^4}\left(\sum_i\omega_i^2\right)^2+\frac{S_p^3}{\pi^4}\sum_i\omega_i^4+\cdots\right], \tag{3.21}$$

where c_p, S_p^1, S_p^2 and S_p^3 are listed in table 1 and we recall that $p = 9-d$ for $D = 10$.

Under type-IIB S-duality we have $\tilde{g}_s = 1/g_s$ and $\tilde{l}_s = l_s g_s^{1/2}$, so that for the type-IIB NS1 and NS5-brane we need $\ell = \tilde{l}_s$ and the thermodynamics is exactly the same as for the D1 and D5-brane when expressed in terms of λ and N. Note that the free energies for the Dp-branes with $p \leq 4$ are negative, while the D5 and NS5-brane have zero free energy and the D6-brane positive free energy.

p	c_p	S_p^1	S_p^2	S_p^3
0	$(2^{21}3^25^77^{-19}\pi^{14})^{1/5}$	$\frac{49}{40}$	$-\frac{1029}{3200}$	$\frac{2401}{1280}$
1	$2^43^{-4}\pi^{5/2}$	$\frac{9}{8}$	$-\frac{27}{128}$	$\frac{81}{64}$
2	$(2^{13}3^55^{13}\pi^8)^{1/3}$	$\frac{25}{24}$	$-\frac{125}{1152}$	$\frac{625}{768}$
3	$2^{-3}\pi^2$	1	0	$\frac{1}{2}$
4	$2^53^{-7}\pi^2$	$\frac{9}{8}$	$\frac{27}{128}$	$\frac{81}{256}$
6	$-2^3\pi^4$	$-\frac{1}{8}$	0	$\frac{5}{256}$

Table 1: Relevant coefficients for the free energy of Dp-branes.

In the discussion above, we have chosen to explicitly write down the free energies in terms of the variables of the dual field theories, since these expressions will play an important role below. Of course, the same can be done for the other thermodynamic quantities listed in (3.4) using (3.16), (3.18) and (3.20). Note also that for the special value of $\kappa = \frac{1}{2}(d-2)$ the temperature vanishes, implying that besides the usual extremal limit describing zero temperature field theory, we also have a limit in which the temperature is zero, accompanied by non-zero R-charges.

4. Stability analysis of near-horizon spinning branes

In this section we analyze the critical behaviour of the near-horizon limit of spinning p-branes, using the thermodynamics obtained in section 3.2. Using the mapping between the supergravity solutions and the dual QFTs, as described in section 3.3, we can find the critical behaviour for the strongly coupled dual field theories with non-zero voltages under the R-symmetry.

Section 4.1 presents a general discussion of boundaries of stability in the grand canonical and canonical ensemble. These two ensembles are then considered in more detail in sections 4.2 and 4.3 respectively. In section 4.4 we finally consider the critical exponents in the two ensembles.

4.1 Boundaries of stability

There are two different settings in which we can study the stability of near-horizon spinning branes. In the first one, to which we refer as the grand canonical ensemble, we imagine the system to be in equilibrium with a reservoir of temperature T and angular velocities Ω_i. Thermodynamic stability then requires negativity of the eigenvalues of the hessian of the Gibbs free energy. In particular, a boundary of stability occurs when the determinant of the hessian is zero or infinite. In the second one, referred to below as the canonical ensemble, we have constant angular momenta J_i and a heat reservoir with temperature T. Stability in this situation demands positivity of the heat capacity C_J and the boundaries of stability occur when this specific heat is zero or infinite.

In the case at hand, we have a system in which the thermodynamic quantities are given in terms of the supergravity variables, so that the boundaries of stability will crucially depend on the change of variables between these two descriptions. We will therefore repeatedly need the determinants of the jacobians, and we define $D_{T\Omega}$ as the determinant $\partial(T,\Omega_1,\Omega_2,\ldots,\Omega_n)/\partial(r_H,l_1,l_2,\ldots,l_n)$ and likewise for D_{TJ}, $D_{S\Omega}$ and D_{SJ}.

In the grand canonical ensemble we need the determinant of the hessian of the Gibbs free energy, which can be written as[13]

$$\det \mathrm{Hes}(-F) = \frac{D_{SJ}}{D_{T\Omega}} \tag{4.1}$$

so that the zeroes of the two determinants D_{SJ}, $D_{T\Omega}$ determine the boundaries of stability. For completeness and use below we also give the specific heat

$$C_\Omega = T\left(\frac{\partial S}{\partial T}\right)_{\Omega_1,\ldots,\Omega_n} = T\frac{D_{S\Omega}}{D_{T\Omega}}\,, \tag{4.2}$$

[13]One could also use the determinant of the hessian of the internal energy $E(S,J)$, which is the inverse of the hessian of the Gibbs free energy.

showing that $\det \mathrm{Hes}(F)$ and C_Ω may have different zeroes. In the canonical ensemble, on the other hand, we need the specific heat

$$C_J = T\left(\frac{\partial S}{\partial T}\right)_{J_1,\dots,J_n} = T\frac{D_{SJ}}{D_{TJ}} \tag{4.3}$$

and the boundaries of stability are determined by the determinants D_{SJ} and D_{TJ}.

In further detail, using the thermodynamic quantities in (3.4) it then follows that

$$\det \mathrm{Hes}(-F) = 8\pi^2\left(\frac{V_p V(S^{d-1})h^{d-2}}{8\pi G}\right)^{n+1} r_H^4 \prod_i (1+x_i)^2 \frac{\Delta_{SJ}}{\Delta_{T\Omega}} \tag{4.4a}$$

$$C_\Omega = \frac{V_p V(S^{d-1})}{4G} h^{\frac{d-2}{2}} r_0^{\frac{d-2}{2}} r_H (d-2-2\kappa)\frac{\Delta_{S\Omega}}{\Delta_{T\Omega}} \tag{4.4b}$$

$$C_J = \frac{V_p V(S^{d-1})}{4G} h^{\frac{d-2}{2}} r_0^{\frac{d-2}{2}} r_H (d-2-2\kappa)\frac{\Delta_{SJ}}{\Delta_{TJ}} \tag{4.4c}$$

where the functions Δ are related to the determinants D up to positive define functions, and given by

$$\begin{aligned}\Delta_{T\Omega} &= (d-4)\Big[d-2-(d-4)\sum_i x_i + (d-6)\sum_{i<j} x_i x_j - \\ &\quad -(d-8)\sum_{i<j<k} x_i x_j x_k + (d-10)x_1x_2x_3x_4\Big]\,, \end{aligned} \tag{4.5a}$$

$$\begin{aligned}\Delta_{S\Omega} &= d-(d-4)\sum_i x_i + (d-8)\sum_{i<j} x_i x_j - \\ &\quad -(d-12)\sum_{i<j<k} x_i x_j x_k + (d-16)x_1x_2x_3x_4\,, \end{aligned} \tag{4.5b}$$

$$\begin{aligned}\Delta_{TJ} &= (d-4)(d-2) - 2(d-8)\sum_i \frac{x_i}{1+x_i} - \\ &\quad -4(d-2)\sum_i \frac{x_i^2}{(1+x_i)^2} + 16\sum_{i<j}\frac{x_i}{1+x_i}\frac{x_j}{1+x_j}\,, \end{aligned} \tag{4.5c}$$

$$\Delta_{SJ} = d\,. \tag{4.5d}$$

Here, we have defined the dimensionless ratios

$$x_i = \frac{l_i^2}{r_H^2}\,. \tag{4.6}$$

The expressions are written for the case of maximal possible number of angular momenta $n=4$, but hold also for $n<4$ by setting the appropriate $x_i = 0$. One observes that, as seen for the free energy in (3.11), the case $d=4$ is special since $\Delta_{T\Omega}=0$ identically, implying that the coordinates $(T,\{\Omega_i\})$ are not independent.

Since $\Delta_{SJ} \neq 0$, the boundaries of stability in the two ensembles can thus be determined as follows: In the grand canonical ensemble a boundary is reached when $\Delta_{T\Omega} = 0$ and the hessian of the Gibbs free energy diverges. In the canonical ensemble on the other hand, we have a boundary of stability when $\Delta_{TJ} = 0$, in which case the specific heat C_J diverges. More precisely, the boundaries of stability are n-dimensional submanifolds in the $(n+1)$-dimensional phase diagram, where one of these two determinants vanish. In the following subsections we study these conditions for the special case of $m \leq n$ equal angular momenta, supplemented with a detailed discussion for the simplest case of one non-zero angular momentum. For the non-dilatonic branes this analysis was performed in refs. [46, 47, 50].

It should be remarked that, at first sight, the analysis shows that we do not have any first-order phase transitions, since all first derivatives of the thermodynamic potentials are continuous everywhere. The phase transitions are instead second-order, though this result should be taken with care, since [50] has given evidence for a first-order phase transition. We will comment on this possibility in the next subsection. Another general result of the analysis is that the boundaries of stability are distinct in the two ensembles that we consider, with a larger region of stability in the canonical ensemble as compared to the grand canonical ensemble, in accordance with standard thermodynamics.

We emphasize here that although we have phrased the analysis in terms of the variables (r_H, x_i), these are in one-to-one correspondence with the thermodynamic extensive variables (S, J_i) through the relations

$$\sqrt{x_i} = \frac{2\pi J_i}{S}, \qquad r_H^{d/2} = \left(\frac{V_p V(S^{d-1})}{16\pi G}\right)^{-1} h^{-(d-2)^2/2} \frac{S}{4\pi} \prod_i \sqrt{1+x_i}\,, \tag{4.7}$$

which follow from (3.4) and (4.6). Hence, conditions on x_i can be directly translated into conditions on the ratio J_i/S. Alternatively, one may rephrase the stability conditions in terms of the dimensionless ratios

$$\chi_i = \frac{E^{d/2}}{J_i^{d-2}} = d^{d/2} 2^{4-3d} \left(\frac{V_p V(S^{d-1})}{16\pi G}\right)^{-(d-4)/2} h^{-(d-2)^2/2} \left(\frac{r_0}{l_i}\right)^{d-2}, \tag{4.8}$$

where

$$\left(\frac{r_0}{l_i}\right)^{d-2} = x_i^{(d-2)/2} \prod_{j=1}^{n} (1+x_j)\,. \tag{4.9}$$

For the case of one angular momentum, χ is up to a numerical constant the variable used in the D3-brane analysis of refs. [46, 50].

4.2 Grand canonical ensemble

We consider the case of $m \leq n$ equal non-zero angular momentum, so that $x_i = x = l^2/r_H^2$, $i = 1, \ldots, m$, in which case the relevant quantity $\Delta_{T\Omega}$ in (4.5a) simplifies to

$$\Delta_{T\Omega} = (d-4)\big(d-2-(d-2-2m)x\big)(1-x)^{m-1} \,. \tag{4.10}$$

We first note that for $d = 3$ (which includes the D6-brane) $\det \mathrm{Hes}(-F)$ is less than zero for all x, and hence corresponds to an unstable situation. The case $d = 4$ (which includes the D5 and NS5-brane), for which T and Ω_i, $i = 1 \ldots n$ are not independent will be treated separately at the end of this subsection, so in the following we assume $d \geq 5$. In this case, we know that for zero angular momentum, i.e. $x = 0$, the branes are stable. We will be concerned only with the first instability that occurs as x is increased, which is hence determined by the first zero of $\Delta_{T\Omega}$.

It follows from (4.10) that there is a boundary of stability at the value

$$x_c^{(m)} = \begin{cases} \dfrac{d-2}{d-4}\,, & m = 1 \\ 1\,, & m > 1 \end{cases} \tag{4.11}$$

In further detail, stability requires $x \leq x_c^{(m)}$ or equivalently, using (4.7) this becomes

$$J \leq \sqrt{x_c^{(m)}} \cdot \frac{S}{2\pi} \tag{4.12}$$

One may also calculate from (3.4) that for m equal angular momenta

$$\tilde{\omega} = \frac{\sqrt{x}}{1 + x/x_\star^{(m)}}\,, \tag{4.13}$$

where we recall the definitions $\tilde{T} = 4\pi T/(d-2)$, $\tilde{\omega} = \Omega/\tilde{T}$ and we have defined

$$x_\star^{(m)} = \frac{d-2}{d-2-2m}\,. \tag{4.14}$$

If for instance $S(T, \{\Omega_i\})$ is known, eq. (4.13) can be viewed as an equation of state using $\sqrt{x} = 2\pi J/S$. With the critical values of x in (4.11) the corresponding critical values of $\tilde{\omega}$ are determined by substitution in (4.13) so that

$$\tilde{\omega}_c^{(m)} = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{d-2}{d-4}}\,, & m = 1 \\ \dfrac{d-2}{2(d-2-m)}\,, & m > 1 \end{cases} \tag{4.15}$$

summarized together with $x_c^{(m)}$ in table 2. As seen from the table, the critical values $\tilde{\omega}_c^{(m)}$ increase as the number of non-zero angular momenta increases, so turning on more equal-valued angular momenta has a stabilizing effect.

		GCE		CE		
d	m	$x_c^{(m)}$	$\tilde{\omega}_c^{(m)}$	$\hat{x}_c^{(m)}$	$\tilde{j}_c^{(m)}$	$x_T^{(m)}$
5	1	3	$\frac{\sqrt{3}}{2}$	$2+\sqrt{5}$	7.238	
	2	1	$\frac{3}{2}$			3
6	1	2	$\frac{\sqrt{2}}{2}$	$\frac{5+\sqrt{33}}{2}$	14.12	
	2	1	1			
	3	1	2			2
7	1	$\frac{5}{3}$	$\frac{\sqrt{15}}{6}$	$\frac{16+\sqrt{301}}{3}$	49.59	
	2	1	$\frac{5}{6}$	$\frac{17}{5}+\frac{2}{5}\sqrt{91}$	165.5	
	3	1	$\frac{5}{4}$			5
8	1	$\frac{3}{2}$	$\frac{\sqrt{6}}{4}$			
	2	1	$\frac{3}{4}$	$3+2\sqrt{3}$	211.5	
	3	1	1			
	4	1	$\frac{3}{2}$			3
9	1	$\frac{7}{5}$	$\frac{\sqrt{35}}{10}$			
	2	1	$\frac{7}{10}$	$\frac{11}{3}+\frac{2}{3}\sqrt{39}$	522.2	
	3	1	$\frac{7}{8}$	$\frac{32}{7}+\frac{3}{7}\sqrt{141}$	4589	
	4	1	$\frac{7}{6}$			7

Table 2: Boundaries of stability in the grand canonical ensemble (GCE) and canonical ensemble (CE) for $m \leq n$ equal non-zero angular momenta. The values in the last column give zero temperature.

Specific heat. It is also interesting to examine the behavior of the specific heat C_Ω in (4.4b), for which we need in addition to (4.10),

$$\Delta_{S\Omega} = \Big(d-(d-4m)x\Big)(1-x)^{m-1}\,,$$
$$(d-2-2\kappa) = \frac{1}{(1+x)^m}[d-2+(d-2-2m)x]\,, \tag{4.16}$$

which follows from (4.5b) and κ in (2.23). Besides a diverging specific heat at $x_c^{(m)}$, we see that C_Ω vanishes, on the other hand, for the values

$$x_0^{(m)} = \frac{d}{d-4m}\,, \qquad x_T^{(m)} = -x_\star^{(m)} \tag{4.17}$$

obtained from the zero of $\Delta_{S\Omega}/\Delta_{T\Omega}$ and the temperature T using (4.16).[14]

[14] Note also that for $d=3$ the specific heat vanishes at $x_T = 1$.

One non-zero angular momentum. In the remainder of this subsection we restrict to the case of one non-zero angular momentum, which by itself exhibits various interesting physical phenomena. The stable region is $x \leq x_c$, with the critical value given by

$$x_c \equiv x_c^{(1)} = \frac{d-2}{d-4} \tag{4.18}$$

or using (4.7),

$$J \leq \sqrt{\frac{d-2}{d-4}}\frac{S}{2\pi}\,. \tag{4.19}$$

The stability requirement thus sets an upper bound on the angular momentum, and a phase transition occurs at the critical value of the angular momentum density. In the dual field theory, this corresponds to a critical value of the R-charge density. From eqs. (3.4) one can also derive the general formulae

$$\tilde{\omega} = \frac{\sqrt{x}}{1+x/x_c} \tag{4.20a}$$

$$\frac{1}{\tilde{T}} = \frac{\sqrt{1+x}}{1+x/x_c} r_H^{(4-d)/2} h^{(d-2)/2} \tag{4.20b}$$

It is not difficult to see that at the boundary of stability $x = x_c$ where the hessian diverges, the ratio $\tilde{\omega}$ is maximized,[15] so that the supergravity description sets an upper bound on this quantity,

$$\tilde{\omega} \leq \tilde{\omega}_c = \frac{1}{2}\sqrt{x_c}\,. \tag{4.21}$$

Moreover, as easily seen from (4.20a), for each value of $\tilde{\omega}$ below this maximum there are two values of x, one corresponding to a stable and the other to an unstable configuration. In particular the two supergravity descriptions with (r_H, x) and $(\tilde{r}_H, \tilde{x})$ related by

$$\tilde{x} = \frac{x_c^2}{x}\,, \qquad \tilde{r}_H = r_H \left(\frac{x^2}{x_c^2}\frac{1+x_c^2/x}{1+x}\right)^{1/(d-4)} \tag{4.22}$$

give the same values of T, Ω. The phase diagram therefore consists of two sheets, a stable one and an unstable one.

As an illustration consider a process in which one starts with a non-rotating non-extremal brane at given $r_0 = r_H$ and turn on the angular momentum l adiabatically, while keeping the horizon radius constant. When the critical value $\tilde{\omega}_c$ is reached

[15] As a consequence, one could have determined this boundary of stability by maximizing Ω/T, providing an alternative method without having to resort to computing jacobians.

the configuration becomes unstable and for the D3-brane two scenarios have been proposed [50]: D-brane fragmentation, in which the branes fly apart in the transverse dimension, and phase mixing in which angular momentum localizes on the brane. The latter possibility will be briefly discussed below for the general p-brane. Note also that for vanishing horizon radius but non-zero angular momentum we have that $x \to \infty$, so that the brane is unstable in this situation, and turning on adiabatically the horizon radius would not cure this instability. Note, however, that C_Ω is positive not only for $x < x_c$ but also for $x > x_0 \equiv x_0^{(1)}$ in (4.17), so the specific heat will be positive in this situation.

Free energy on the two branches. It is possible to obtain a closed form expression for the free energy on the two branches $x \leq x_c$ and $x \geq x_c$ respectively. To this end we solve (4.20a) for x yielding the two solutions

$$x_\pm = 8\frac{\tilde{\omega}_c^4}{\tilde{\omega}^2}\left(1 - \frac{1}{2}\left(\frac{\tilde{\omega}}{\tilde{\omega}_c}\right)^2 \pm \sqrt{1 - \left(\frac{\tilde{\omega}}{\tilde{\omega}_c}\right)^2}\right), \tag{4.23}$$

where we have used the value of $\tilde{\omega}_c$ in (4.21). It is easy to check that the solution x_- has the property that $x_- \to 0$ when $\omega \to 0$, whereas the other solution x_+ goes to infinity in that limit. Thus, x_- describes the stable branch $0 < x \leq x_c$ and x_+ the unstable branch $x > x_c$. To obtain the explicit expression for the free energy we use (4.20b) to express r_H in terms of T and x, as well as (3.11), which together with the horizon equation (2.8) implies $F \sim (1+x)r_H^{d-2}$. The resulting free energy for each of the two branches is then

$$F_\pm = -\frac{V_p V(S^{d-1})}{16\pi G} h^{\frac{(d-2)^2}{(d-4)}} \frac{d-4}{2} \tilde{T}^{\frac{(2d-4)}{(d-4)}} (1+x_\pm)^{\frac{2(d-3)}{(d-4)}} \left(1 + \frac{x_\pm}{x_c}\right)^{-2\frac{d-2}{d-4}}, \tag{4.24}$$

with $x_\pm$ given in (4.23). As a check, we note that expanding F_- for small $\tilde{\omega}$ reproduces the expansion given in (3.14), as it should. Differentiating F_- with respect to T gives the entropy $S(T, \Omega)$, so that we can use $\sqrt{x} = 2\pi J/S$ in (4.20a) to determine the exact form of the equation of state for one non-zero angular momentum. As a curiosity we also mention the expansion of the free energy F_+ on the unstable branch,

$$F_+ \sim T^2 \Omega^{4/(d-4)} \left[1 + \mathcal{O}\left(\frac{\Omega}{T}\right)\right] \tag{4.25}$$

exhibiting a universal T^2 dependence for all branes, but due to the unstable nature of this branch the relevance of this expression is presently unclear.

Phase mixing. In ref. [50], it was shown that for the spinning D3-brane there exists a possibility that a mixing of these two phases is thermodynamically favored (maximizing entropy), so that as a consequence of the instability angular momentum is localized on the brane. To carry out this analysis for the general case $d > 4$, one needs to work in the microcanonical ensemble and consider the mixed states determined by (4.22). Thus the problem is to maximize the entropy

$$S_{\rm av} = \mu S(r_H, x) + (1-\mu) S(\tilde{r}_H, \tilde{x}) \,, \tag{4.26}$$

for given energy $E_{\rm av}$ and angular momentum $J_{\rm av}$, subject to the constraints

$$E_{\rm av} = \mu E(r_H, x) + (1-\mu) E(\tilde{r}_H, \tilde{x}) \,, \qquad J_{\rm av} = \mu J(r_H, x) + (1-\mu) J(\tilde{r}_H, \tilde{x}) \,, \tag{4.27}$$

with $(\tilde{r}_H, \tilde{x})$ expressed in (r_H, x) through (4.22). We have not carried out this analysis but expect that the features observed for $d = 6$ (including a first-order phase transition) in [50], will persist for the other cases $d > 4$. We thus expect that there will be a mixed state and first-order phase transition at some critical value of $x < x_c$.

The case $d = 4$. As pointed out before, the case $d = 4$ needs a special treatment, since from (3.11) we have that the free energy vanishes, so that $dF = 0$. This is due to the fact that the $n + 1$ variables $(T, \{\Omega_i\})$ are not independent anymore. Indeed, for one non-zero angular momentum we read off from (3.4) that

$$(2\pi T)^2 + \Omega^2 = h^{-2} \,, \tag{4.28}$$

which characterizes the phase space. For zero angular momentum we recover the known fact that the temperature is constant for the NS5 and D5-branes. As a further check, using also $\Omega/T = (2\pi)^2 J/S$ in this case, the curve (4.28) implies that $SdT + Jd\Omega = 0$, in accord with the thermodynamic relation (3.12) with $dF = 0$.

4.3 Canonical ensemble

We also discuss the canonical ensemble for m non-zero equal angular momenta, for which the relevant quantity Δ_{TJ} takes the form,

$$\begin{aligned}\Delta_{TJ} = \frac{1}{(1+x)^2} \Big[&(d-2)(d-4) + 2\Big((d-2)(d-4) - (d-8)m\Big)x + \\ &+ (d-2-2m)(d-4-4m)x^2 \Big]\end{aligned} \tag{4.29}$$

and we recall that the specific heat C_J also vanishes at the zeroes of the temperature, i.e. $x_T^{(m)}$ given in (4.17). The case $d = 4$ is stable for any m and for $d = 3$ we find the curious behavior that there is lower bound on x namely $(-4 + \sqrt{21})/5$, so that e.g. the non-rotating D6-brane is unstable, but becomes stable in the canonical ensemble when the angular momentum is large enough. In the following we will restrict again to $d \geq 5$.

The positive solutions of the quadratic equation (4.29) are listed as $\hat{x}_c^{(m)}$ in table 2, and correspond to the boundaries of stability, with the property that for $x \leq \hat{x}_c^{(m)}$ the branes are stable. From the table we infer a number of observations: For one non-zero angular momentum the branes with $d = 8, 9$ are stable for any value of x, but when more angular momenta are switched on a boundary of stability emerges. Moreover, for maximal number of non-zero angular momenta all branes are stable.

To further examine the boundary of stability we use (3.4) to construct the dimensionless ratio

$$\tilde{j} = \frac{J^{d-4}}{\tilde{T}^d \zeta_d} = (2\sqrt{x})^{d-4}(1+x)^{d-2m}\left(1 + \frac{x}{x_\star^{(m)}}\right)^{-d} \tag{4.30a}$$

$$\zeta_d \equiv \left(\frac{V_p V(S^{d-1})}{16\pi G}\right)^{d-4} h^{(d-2)^2}\,, \tag{4.30b}$$

where $x_\star^{(m)}$ is defined in (4.14). The numerical values of the relevant ratio $\tilde{j}_c$ on the boundary are also listed in the table 2. Note that, in analogy with the quantity $\tilde{\omega}$ relevant for the grand canonical ensemble, in this case the boundary of stability occurs also precisely at the maximum of the ratio $\tilde{j}$ in (4.30a). One can also easily obtain the corresponding critical values of $\tilde{\omega}$ using (4.13) and the critical values $\hat{x}_c^{(m)}$. In parallel with the grand canonical ensemble the phase diagram for the cases with a boundary of stability consists again of a stable and unstable sheet. It would be interesting to examine the possibility of phase mixing along the lines described in the previous subsection.

4.4 Critical exponents

We conclude this section with a general analysis of the critical exponents in both the ensembles for the case of one non-zero angular momentum. To this end, we note that besides the specific heats (4.4b) and (4.4c), one also has the response functions

$$\begin{aligned}
\chi_T &= \left(\frac{\partial J}{\partial \Omega}\right)_T = \frac{V_p V(S^{d-1}) h^{d-2}}{8\pi G} r_H^2 (1+x)^2 \frac{\Delta_{TJ}}{\Delta_{T\Omega}}\,, \\
\alpha_\Omega &= \left(\frac{\partial J}{\partial T}\right)_\Omega = \frac{V_p V(S^{d-1}) h^{d-2}}{G} r_H^2 \sqrt{x}\, \frac{2-(d-4)x}{\Delta_{T\Omega}} \\
\alpha_J &= \left(\frac{\partial \Omega}{\partial T}\right)_J = -\frac{8\pi\sqrt{x}}{(1+x)^2} \frac{2-(d-4)x}{\Delta_{TJ}}\,,
\end{aligned}$$

where χ_T is the isothermal capacitance. The following discussion pertains to the cases in which a boundary of stability was found in the one angular momentum case, i.e. $d \geq 5$ in the grand canonical ensemble, and $d = 3, 5, 6, 7$ in the canonical ensemble.

Starting with the grand canonical ensemble, we consider a point (T_c, Ω_c) on the boundary of stability.[16] Following a similar analysis as in [47], we show that this point behaves as a critical point in ordinary thermodynamics. The stable region has $T \geq T_c$ and $\Omega \leq \Omega_c$, so we define the quantities

$$\epsilon_T = \frac{T - T_c}{T_c}, \qquad \epsilon_\Omega = \frac{\Omega_c - \Omega}{\Omega_c} \tag{4.31}$$

and consider a function $f(T, \Omega)$ near the point (T_c, Ω_c). The critical exponents n_T and n_Ω for $f(T, \Omega)$ are then defined as

$$n_T = -\lim_{\epsilon_T \to 0} \left. \frac{\ln f}{\ln \epsilon_T} \right|_{\epsilon_\Omega = 0} = -\lim_{\epsilon_T \to 0} \frac{d \ln f|_{\epsilon_\Omega = 0}}{d \ln \epsilon_T}, \tag{4.32a}$$

$$n_\Omega = -\lim_{\epsilon_\Omega \to 0} \left. \frac{\ln f}{\ln \epsilon_\Omega} \right|_{\epsilon_T = 0} = -\lim_{\epsilon_\Omega \to 0} \frac{d \ln f|_{\epsilon_T = 0}}{d \ln \epsilon_\Omega}. \tag{4.32b}$$

We assume that the function satisfies

$$f(T, \Omega_c) = \frac{g(T)}{h_T(x)}, \qquad f(T_c, \Omega) = \frac{\tilde{g}(\Omega)}{h_\Omega(x)}, \tag{4.33}$$

where $g(T_c)$ and $\tilde{g}(\Omega_c)$ are finite and different from zero and both h_T and h_Ω satisfy $h|_{x=x_c} = 0$ and $dh/dx|_{x=x_c} \neq 0$. This is indeed true for the response functions C_Ω, χ_T, α_Ω and the quantities $(S - S_c)^{-1}$ and $(J - J_c)^{-1}$.

We first approach the critical point by varying the temperature, and hence put $\Omega = \Omega_c$. Using (4.20a) and (4.21) one obtains

$$\epsilon_T(x) = \frac{1}{2}\sqrt{\frac{x_c}{x}}\left(1 + \frac{x}{x_c}\right) - 1 \tag{4.34}$$

and substituting in (4.32a) one finds

$$\begin{aligned} n_T &= -\lim_{\epsilon_T \to 0} \frac{d \ln f|_{\epsilon_T = 0}}{d \ln \epsilon_T} = -\lim_{\epsilon_T \to 0} \frac{1}{f} \frac{df}{d \ln \epsilon_T} = -\lim_{x \to x_c} \frac{\epsilon_T}{f} \frac{df}{dx} \left(\frac{d\epsilon_T}{dx}\right)^{-1} \\ &= -\lim_{x \to x_c} \frac{\epsilon_T}{h_T} \frac{dh_T}{dx} \left(\frac{d\epsilon_T}{dx}\right)^{-1} = \frac{1}{2}. \end{aligned} \tag{4.35}$$

Here, the last step follows from $h_T|_{x=x_c} = 0$, $dh_T/dx|_{x=x_c} \neq 0$, $d\epsilon_T/dx|_{x=x_c} = 0$ and $d^2\epsilon_T/dx^2|_{x=x_c} \neq 0$. On the other hand, approaching the critical line by varying Ω, we need to put $T = T_c$ and have

$$\epsilon_\Omega(x) = 1 - 2\sqrt{\frac{x}{x_c}} \frac{1}{1 + x/x_c}. \tag{4.36}$$

Since $d\epsilon_\Omega/dx|_{x=x_c} = 0$ and $d^2\epsilon_\Omega/dx^2|_{x=x_c} \neq 0$ we also find that $n_\Omega = 1/2$.

[16] The boundary of stability does not have any special point other than $(T = 0, \Omega = 0)$ so we take a generic point different from that.

Since each of the functions C_Ω, χ_T, α_Ω, $(S - S_c)^{-1}$ and $(J - J_c)^{-1}$ is of the form (4.33), one immediately concludes that for each of these quantities the critical exponents is equal to 1/2. The common value 1/2, which was earlier found [47] for the non-dilatonic branes,[17] apparently persists and this value has been shown to be in agreement with scaling laws in statistical physics [47].

The critical analysis in the canonical ensemble proceeds along the same lines. In this case we consider a point (T_c, J_c) on the boundary of stability. Repeating the analysis above essentially with the replacement $\Omega \to J$, and using the fact that C_J, α_J, $(S - S_c)^{-1}$ and $(\Omega - \Omega_c)^{-1}$ are all of the form (4.33) (with $\Omega \to J$), it follows that also in this case all critical exponents are 1/2.

5. Field theory analysis

In this section we consider the quantum field theories living on the D and M-branes in the limit where they are free field theories. Using the ideal gas approximation we compute in section 5.1 the free energies with non-zero R-voltage under the R-symmetry. We do this to compare the thermodynamic behaviour in the weak coupling limit with the strong coupling limit. In section 5.2 we discuss the interpolation between weak and strong coupling, while in section 5.3 we find the stability behaviour at weak coupling and compare this to the strong coupling limit.

5.1 The free energy for weakly coupled field theory

In this section we calculate the free energies for the extremely weakly coupled limit of the dual field theories extending the regularization method used in [46, 50] for the D3, M2 and M5-brane.

We start by writing the free energy with all R-charge voltages $\{\Omega_i\}$ turned off. As we shall see, this depends only on the spatial dimension p of the field theory, and on the number of massless bosonic and fermionic degrees of freedom. In particular, the field theories that we consider have 16 supercharges so that for $N = 1$ these theories have 8 bosonic and 8 fermionic degrees of freedom. Using the ideal gas approximation, where particles are assumed to have negligible interaction, we get the free energy[18]

$$F = TV_p \int \frac{d^p q}{(2\pi)^p} \left[8 \log \left(1 - e^{-\beta |q|} \right) - 8 \log \left(1 + e^{-\beta |q|} \right) \right] = -k_p V_p T^{p+1} \,, \qquad (5.1)$$

with

$$k_p = 2^{4-p} (2 - 2^{-p}) \frac{(p-1)!}{\Gamma(p/2) \pi^{p/2}} \zeta(p+1) \,, \qquad (5.2)$$

where $p \geq 1$.

[17] These critical exponents should be related to the corresponding exponents in correlation functions of the field theory. In the field theory analysis for the D3-brane case [46] no agreement was found, but a mean field treatment was suggested to cure this discrepancy.

[18] For a confining theory, it is understood that the temperature is above the confining temperature.

JHEP01(2000)008

n	Bosons	Fermions
1	$6(0),\ (\pm 1)$	$4(\pm\frac{1}{2})$
2	$4(0),\ (\pm 1,0),\ (0,\pm 1)$	$2(\pm\frac{1}{2},\pm\frac{1}{2})$
3	$2(0),\ (\pm 1,0,0),\ (0,\pm 1,0),\ (0,0,\pm 1)$	$(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$
4	$(\pm 1,0,0,0),\ (0,\pm 1,0,0),$ $(0,0,\pm 1,0),\ (0,0,0,\pm 1)$	$(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ number of pluses = even

Table 3: Weights for the 8 bosons and 8 fermions in the four possible cases labeled by $n = [d/2]$, corresponding to $3 \leq d \leq 8$. Numbers in front of the weights denote the degeneracy of the spectrum with respect to this weight. In the $n = 4$ case the 8 fermions all have same chirality under the SO(8).

If we consider non-zero R-voltage, we must replace $\beta|q|$ with $\beta|q| + \beta\sum_{i=1}^{n}\alpha_i\Omega_i$ in the partition function, where $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the SO(d) weight vector of the particle. The resulting free energy is

$$F = TV_p \int \frac{d^p q}{(2\pi)^p} \sum_{\vec{\alpha}} s_{\vec{\alpha}} \log\left[1 - s_{\vec{\alpha}} \exp\Big(-\beta|q| - \beta\sum_{i=1}^{n}\alpha_i\Omega_i\Big)\right] \qquad (5.3)$$

where $\vec{\alpha}$ runs over the 16 different particles and $s_{\vec{\alpha}}$ is $+1$ for bosons and -1 for fermions. The weights $\vec{\alpha}$ for the different SO(d) R-charge groups are listed in table 3. Note that the weights for SO($2n$) and SO($2n+1$) are the same, and that branes with identical R-symmetry group have the same weights.

The integrals for the 8 bosons in (5.3) are clearly divergent since $\beta\Omega_i$ is real. In ref. [46] it was proposed to perform an analytic continuation by considering $\beta\Omega_i$ to be complex, so that using (E.5) the free energies (5.3) can be expressed in terms of polylogarithms,

$$F = -\frac{\Gamma(p)}{2^{p-1}\pi^{p/2}\Gamma(p/2)} V_p T^{p+1} \sum_{\vec{\alpha}} \mathrm{Li}_{p+1}\left[s_{\vec{\alpha}} \exp\Big(-\sum_{i=1}^{n}\alpha_i\omega_i\Big)\right], \qquad (5.4)$$

where $\omega_i = \beta\Omega_i$. The polylogarithms are not defined for real numbers greater than one, but in appendix E we discuss the continuation to this region, along with some general properties of polylogarithms.

Using the exact functions $B_n(x)$ and $F_n(x)$ for $x \in \mathbb{R}$ of appendix E, we can in principle write all the free energies for the different p-branes exactly. To save space, we restrict ourselves to write the energies with odd p exactly, and write the energies with even p to fourth order in ω_i. In section 5.3, however, we use the fact that all the free energies are known to all orders in ω_i.

For the M-branes, it is believed that $N = 1$ corresponds to a free field theory, while for $N > 1$ the field theories are interacting. Thus, for a single M2-brane and

M5-brane we have

$$F_{\rm M2} = -V_2T^3\frac{1}{\pi}\left[7\zeta(3)-\frac{1}{2}\sum_{i=1}^{4}\log(\omega_i)\omega_i^2+\left(\frac{1}{2}\log(2)+\frac{3}{4}\right)\sum_{i=1}^{4}\omega_i^2+\right.$$
$$\left.+\frac{1}{128}\left(\sum_{i=1}^{4}\omega_i^2\right)^2-\frac{10}{1152}\sum_{i=1}^{4}\omega_i^4+\frac{1}{16}\omega_1\omega_2\omega_3\omega_4+\mathcal{O}(\omega_i^6)\right], \quad (5.5a)$$

$$F_{\rm M5} = -V_5T^6\left[\frac{\pi^3}{30}+\frac{\pi}{24}(\omega_1^2+\omega_2^2)+\frac{1}{96\pi}(\omega_1^2+\omega_2^2)^2+\frac{1}{48\pi}(\omega_1^4+\omega_2^4)+\right.$$
$$\left.+\frac{1}{1152\pi^3}(\omega_1^2+\omega_2^2)^3-\frac{1}{288\pi^3}(\omega_1^6+\omega_2^6)\right]. \quad (5.5b)$$

The ideal gas approximation is valid for the D-branes when $\lambda = 0$. In this limit, the free energies for N Dp-branes take the form

$$F_{\rm D1} = -2\pi N^2V_1T^2 \quad (5.6a)$$

$$F_{\rm D2} = -N^2V_2T^3\frac{1}{\pi}\left[7\zeta(3)-\frac{1}{2}\sum_{i=1}^{3}\log(\omega_i)\omega_i^2+\left(\frac{1}{2}\log(2)+\frac{3}{4}\right)\sum_{i=1}^{3}\omega_i^2+\right.$$
$$\left.+\frac{1}{128}\left(\sum_{i=1}^{3}\omega_i^2\right)^2-\frac{10}{1152}\sum_{i=1}^{3}\omega_i^4+\mathcal{O}(\omega_i^6)\right], \quad (5.6b)$$

$$F_{\rm D3} = -N^2V_3T^4\left[\frac{\pi^2}{6}+\frac{1}{4}\sum_{i=1}^{3}\omega_i^2+\frac{1}{32\pi^2}\left(\sum_{i=1}^{3}\omega_i^2\right)^2-\frac{1}{16\pi^2}\sum_{i=1}^{3}\omega_i^4\right], \quad (5.6c)$$

$$F_{\rm D4} = -N^2V_4T^5\frac{1}{\pi^2}\left[\frac{93}{8}\zeta(5)+\frac{21}{16}\zeta(3)(\omega_1^2+\omega_2^2)+\left(\frac{25}{192}+\frac{1}{64}\log(2)\right)(\omega_1^4+\omega_2^4)+\right.$$
$$\left.+\frac{3}{32}\log(2)\omega_1^2\omega_2^2+\frac{1}{16}\Big(-\log(\omega_1)\omega_1^4-\log(\omega_2)\omega_2^4\Big)+\mathcal{O}(\omega_i^6)\right], \quad (5.6d)$$

$$F_{\rm D5} = -N^2V_5T^6\left[\frac{\pi^3}{30}+\frac{\pi}{24}(\omega_1^2+\omega_2^2)+\frac{1}{96\pi}(\omega_1^2+\omega_2^2)^2+\frac{1}{48\pi}(\omega_1^4+\omega_2^4)+\right.$$
$$\left.+\frac{1}{1152\pi^3}(\omega_1^2+\omega_2^2)^3-\frac{1}{288\pi^3}(\omega_1^6+\omega_2^6)\right] \quad (5.6e)$$

$$F_{\rm D6} = -N^2V_6T^7\frac{1}{\pi^3}\left[\frac{1905}{64}\zeta(7)+\frac{465}{128}\zeta(5)\omega^2+\frac{95}{512}\zeta(3)\omega^4+\mathcal{O}(\omega^5)\right]. \quad (5.6f)$$

5.2 Interpolation between weakly and strongly coupled theories

While the expressions (5.5) represent the free energies of the M-branes for $N = 1$, the supergravity results (3.17) and (3.19) are the corresponding free energies in the $N \to \infty$ limit. As discussed in [52] (without R-voltage Ω_i) it is expected that there is a smooth interpolating function $f(N, \{\omega_i\})$ so that the free energy for all N is given by

$$F_N(T, \{\Omega_i\}) = f(N, \{\omega_i\}) F_{N=1}(T, \{\Omega_i\}) \,, \tag{5.7}$$

where $F_{N=1}(T, \{\Omega_i\})$ is the free energy for $N = 1$, given in (5.5). In other words, one can conjecture that if all higher derivative terms in the effective 11-dimensional supergravity action were known, and if one could find the spinning black M-brane solution in this effective action, one could compute the free energies for all N, and in particular the free energies (5.5) for $N = 1$. The status of this conjecture, however, is not clear since there could very well be a phase transition obstructing the smooth interpolation to the free theory limit.

Turning to the D-branes, the expressions in (5.6) represent the free energies for $N \gg 1$ and $\lambda = 0$ (which in particular means that $\lambda \ll r_H^{3-p}$), while the supergravity results (3.21) are valid for $N \gg 1$ and large 't Hooft coupling, $\lambda \gg r_H^{3-p}$ (for the Dp-branes with $p \neq 3$ there is also has an upper bound on λ, see eq. (3.15)). Thus, we can consider the free energies of D-branes with N fixed but with λ varying between the two limits just described. One can then conjecture that for fixed $N \gg 1$ there exists a smooth interpolation function $f(\lambda, T, \{\Omega_i\})$ so that we can write

$$F_\lambda(T, \{\Omega_i\}) = f(\lambda, T, \{\Omega_i\}) F_{\lambda=0}(T, \{\Omega_i\}) \tag{5.8}$$

where $F_{\lambda=0}(T, \{\Omega_i\})$ is the free energy for $\lambda = 0$. Moreover, for the D3-brane the field theory is conformal, so that the function is expected to depend on dimensionless quantities only, i.e. $f(\lambda, \{\omega_i\})$. The possibility of such a smooth interpolation has previously been discussed in [52] for the D3-brane without the R-voltage. An important first check of this conjecture is the fact that the free energies of the D-branes in the two limits show the same N^2 factor in front. This implies that only string loop corrections, which carry factors of $1/N^2$ would modify this behavior, and thus do not have to be considered in the large-N limit. Comparing the λ-dependence on the other hand, one sees that only for the D3-brane the same form is found in the two limits. Again, if one knew the higher derivative terms of the effective action of type-II string theory and one could solve the equations of motion for a spinning black D-brane, one could presumably find the smooth interpolation between the two limits of λ. Again, this conjecture assumes that there is not a phase transition between weak and strong coupling. This assumption has been challenged in [60] for the D3-brane.

We will return to this issue in section 6.2, where we calculate the leading order correction in $\lambda^{-3/2}$ to the D3-brane free energy, due to the $l_s^6 R^4$ term in the type-IIB effective action. In section 5.3 we take another path and compare the thermodynamic behavior of the field theory in the two limits, including a study of the thermodynamic stability.

5.3 Stability behavior at weak coupling

We analyze the thermodynamic stability of the weakly coupled QFTs using the free energies (5.5) and (5.6). For simplicity, we restrict ourselves to the case of one non-zero voltage $\Omega_1 = \Omega$. As a consequence, branes with equal number of spatial dimensions p have the same thermodynamics, since our analysis is not affected by the overall dependence on N.

From the Gibbs free energy $F = F(T,\Omega)$ we compute the heat capacity

$$C_\Omega = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_\Omega \tag{5.9}$$

and in the cases we consider, one can check that C_Ω is always positive. Instead we can extract the stability behaviour by considering the hessian matrix

$$\mathrm{Hes}(F) = \begin{pmatrix} \frac{\partial^2 F}{\partial T^2} & \frac{\partial^2 F}{\partial T \partial \Omega} \\ \frac{\partial^2 F}{\partial T \partial \Omega} & \frac{\partial^2 F}{\partial \Omega^2} \end{pmatrix} \tag{5.10}$$

of the free energy $F = F(T,\Omega)$. For a stable point, the hessian (5.10) should be negative definite. Since C_Ω is positive for $\Omega = 0$, the boundary of stability is reached when one of the eigenvalues of the hessian (5.10) changes sign. Since there are no singularities this occurs when $\det(\mathrm{Hes}(F)) = 0$, so that the boundary of stability is characterized by a certain critical value of $\omega = \Omega/T$. The results of the analysis for the various values of p are given in table 4. To analyze the cases of even p, we use that we know the free energy to all orders in ω. Thus, one can take an appropriate number of terms in order to ensure that the value of ω that one finds has the required accuracy.

p	ω_c
1	Stable
2	1.5404
3	2.4713
4	3.3131
5	4.1458
6	4.9948

Table 4: The boundary of stability for the various p-branes in the weakly coupled field theory limit.

Considering table 4 we see that most branes have a boundary of stability at a certain value of ω, as also seen in the stability analysis of section 4.2. We also remark that all the values of ω in table 4 have the same orders of magnitude as the values in table 2, and thus it seems plausible that the conjectured interpolation between the two limits of the QFTs described in section 5.1 should connect the values of ω. Table 5 summarizes the values of ω for the weak and strong coupling limits of the D, and M-branes, and for the D2, D3, D4, M2 and M5-branes the critical values of ω in the two limits are seen to be remarkably close. We also note that ω is increasing with p in both limits.

The D1-brane and D6-brane, however, are seen from table 5 to have qualitatively different stability behaviour in the weak and strong coupling limits, so that in this case the interpolation should somehow create or destroy a boundary of stability at some special point between the two limits. If we for definiteness think about the QFT living on N D1-branes on top of each other, we see that for $\lambda = 0$ it is stable, also with R-voltage turned on, while for λ large it should exhibit a boundary of stability. Thus, at some value of λ the QFT makes a transition from being completely stable to being potentially unstable. It would be interesting to study how this mechanism works in detail.

For the D5-branes we also have completely different qualitative behaviour. At weak coupling we can vary the thermodynamic parameters T and $\{\Omega_i\}$ freely, while at strong coupling, they are constrained (see section 4.2). Thus, somehow the phase space must expand as one moves away from strong coupling. One could also try to study this phenomenon for non-rotating branes, here the temperature is constant at strong coupling.

Brane	ω_{weak}	ω_{strong}
D1	Stable	1.2825
D2	1.5404	1.6223
D3	2.4713	2.2214
D4	3.3131	3.6276
D5	4.1458	Not defined
D6	4.9948	Unstable
M2	1.5404	1.2825
M5	4.1458	3.6276

Table 5: Comparison between the boundaries of stability for the type-II Dp-branes in the weak and strong coupling limits of λ and for the M-branes in the $N = 1$ and $N \to \infty$ limits.

If we instead consider the canonical ensemble, with variables T and J, one must consider the heat capacity

$$C_J = T\left(\frac{\partial S}{\partial T}\right)_J = T\det(\mathrm{Hes}(F))\left[\left(\frac{\partial J}{\partial \Omega}\right)_T\right]^{-1} \tag{5.11}$$

Thus, we see that C_J is zero whenever $\det(\mathrm{Hes}(F))$ is, i.e. the canonical ensemble has the same stability behaviour as the grand canonical ensemble. This result should not be surprising since we in fact have used standard statistical physics to derive our thermodynamic relations, so that it is expected that general results, such as the equivalence of ensembles, should hold. Nevertheless, it would be interesting to test this by computing corrections to the stability behaviour from the weakly coupled field theory side, or the supergravity side, since the results of section 4 show that the thermodynamic ensembles are not equivalent in the strongly coupled large-N limit. If we consider a D-brane, the expectation would be that the boundaries of stability in the two different ensembles start for $\lambda = 0$ at the same value, move away from each other as λ increases and finally reach the values given in section 4.

Finally we note that, with respect to the critical exponents there is also a qualitative difference between the weak and strong coupling limit of the QFT. As discussed in section 4.4 the heat capacities C_Ω and C_J both behave as $1/\sqrt{T - T_c}$ near the boundary of stability. But, in the weak coupling limit one can easily check that C_Ω and C_J both behave as $(T - T_c)^\alpha$ with $\alpha \geq 0$, α being different for the two heat capacities. Another way to see this, is to note that while the heat capacities in the strong coupling limit have singularities on the boundary of stability, they are continuous in the weak coupling limit.

In conclusion we see that there are many similarities between the thermodynamics at weak and at strong coupling (or small and large N for the M-branes), but also important qualitative differences that are non-trivial to connect by an interpolation between the two limits. In the next section we make the first step towards a quantitative understanding of this conjectured interpolation for spinning branes.

6. Free energies from the supergravity action

It is well known that one can obtain the free energy of a field theory by Wick rotation and evaluating the path integral partition function with the boundary condition that time is periodic with the inverse temperature as period. For general relativity, this method has been applied in order to compute the free energy of a black hole, but, with some difficulty since one need to think of ways to circumvent the problems of quantizing the gravitational field. In Anti-de Sitter space though, this has proved surprisingly easy since one can just evaluate the action on the background geometry [51]. Moreover, the free energy for the near-horizon limit of the D3, M2 and M5-brane has been reproduced with this method [29], which is not surprising since these branes all have Anti-de Sitter space times a sphere as their near-horizon geometry.

In section 6.1 we will extend these results to spinning branes in the near-horizon limit, showing that we are able to reproduce the free energy (3.11) found in section 3.2. In section 6.2 we compute the first correction in $1/\lambda$ to the free energy of the spinning D3-brane found in section 6.1. This will then be used to test the conjecture that there exists a smooth interpolating function between $\lambda = 0$ and $\lambda = \infty$, as discussed in section 5.2.

6.1 Free energies from the low-energy effective action

The euclidean low-energy supergravity action is

$$I_E = I_E^{\rm bulk} + I_E^{\rm bd}\,, \tag{6.1}$$

where the bulk term is given by

$$I_E^{\rm bulk} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^D x\sqrt{g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2(p+2)!}e^{a\phi}F_{p+2}^2\right) \tag{6.2}$$

and the boundary term is [61]

$$I_E^{\rm bd} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{h}K\,, \tag{6.3}$$

with $h_{\mu\nu}$ the boundary metric and K the extrinsic curvature. In this section we obtain the free energy (3.11) as I_E/β, where $\beta = 1/T$ is the inverse temperature and I_E is the regularized value of the euclidean action (6.1) evaluated on the Wick rotated spinning p-brane solution in the near-horizon limit. We restrict ourselves to one non-zero angular momentum, since we expect that because the free energy (3.11) is independent of the angular momentum, more non-zero angular momenta will not alter our final result.

In appendix C we perform a Wick rotation of the near-horizon solution (3.2) in the presence of one non-zero angular momentum, to obtain the euclidean spinning brane solution (C.3). Starting with the bulk term (6.2), we substitute (C.3) and integrate over the time τ and the angles to arrive at the following general expression

$$\frac{L(r)}{\beta} = \frac{V_p V(S^{d-1})}{16\pi G} \frac{(d-2)^3}{D-2} r^{d-3} \left[1 + \sum_{s=1}^{\infty} \left(\tilde{v}_s + \tilde{w}_s \left(\frac{r_0}{r} \right)^{d-2} \right) \left(\frac{\tilde{l}}{r} \right)^{2s} \right] , \tag{6.4}$$

where the $\beta = 1/T$ factor is the period of the time τ. Here the coefficients $\tilde{v}$ and $\tilde{w}$ can be computed in principle through any desired order for any brane solution.

To evaluate the final integral over r we need to introduce a regularization method along the lines of [29, 52]. In this prescription we first perform the integral up to a cutoff radius $r_{\max}$ and subtract the contribution of the extremal brane with a temperature equal to the original brane at the cutoff radius $r_{\max}$. Thus, integrating the expression (6.4) from the horizon radius r_H to the cutoff radius $r_{\max}$ we arrive at

$$\frac{I_E^{\text{bulk}}}{\beta} = \frac{V_p V(S^{d-1})}{16\pi G} \frac{(d-2)^2}{D-2} r^{d-2} \left[1 + \sum_{s=1}^{\infty} \left(v_s + w_s \left(\frac{r_0}{r} \right)^{d-2} \right) \left(\frac{\tilde{l}}{r} \right)^{2s} \right] \Bigg|_{r_H}^{r_{\max}} \tag{6.5}$$

In particular, for the near-horizon spinning solutions in 10-dimensional type II string theory and 11-dimensional M-theory, one finds by explicit evaluation that the expansion coefficients v_s, w_s can be uniformly written[19] as

$$v_1 = -\frac{d-2}{d} , \qquad v_s = -\frac{4}{(2s+d-4)(2s+d-2)} , \qquad s \geq 2 ,$$
$$w_s = -\frac{2}{2s+d-2} , \qquad s \geq 1 , \tag{6.6}$$

with d the transverse dimension. Note that these coefficients satisfy the recursive relations

$$v_s = w_{s-1} - w_s , \qquad v_0 = 1 , \qquad w_0 = -1 \tag{6.7}$$

[19] We have checked these relations up to fourth order in $(\tilde{l}/r)^2$ for all 10-dimensional and 11-dimensional brane solutions, and we believe that they are generally valid. Using these values one can also write a closed form expression for (6.5) in terms of logarithms.

the importance of which will become apparent below. Continuing with (6.5) we find after some algebra that

$$\frac{I_E^{\rm bulk}}{\beta} = \frac{V_p V(S^{d-1})}{16\pi G}\frac{(d-2)^2}{D-2}\left[r_{\rm max}^{d-2} - r_0^{d-2} + \sum_{s=1}^{\infty}\left(v_s r_{\rm max}^{d-2} + w_s r_0^{d-2}\right)\left(\frac{\tilde{l}}{r_{\rm max}}\right)^{2s} - \right.$$
$$\left. - \sum_{s=0}^{\infty}(v_s r_H^{d-2} + w_s r_0^{d-2})\left(\frac{\tilde{l}}{r_H}\right)^{2s}\right]. \tag{6.8}$$

Using the relation (2.8) to write $r_0^{d-2} = r_H^{d-2}(1-(\tilde{l}/r_H)^2)$ the recursion relation (6.7) implies that the last term in (6.8) cancels, giving

$$\frac{I_E^{\rm bulk}}{\beta} = \frac{V_p V(S^{d-1})}{16\pi G}\frac{(d-2)^2}{D-2}\left[r_{\rm max}^{d-2} - r_0^{d-2} + \sum_{s=1}^{\infty}(v_s r_{\rm max}^{d-2} + w_s r_0^{d-2})\left(\frac{\tilde{l}}{r_{\rm max}}\right)^{2s}\right]. \tag{6.9}$$

The regularized bulk contribution to the free energy is

$$F_{\rm bulk} = \lim_{r_{\rm max}\to\infty}\left[\frac{I_E^{\rm bulk}}{\beta} - \left.\frac{I_E^{\rm bulk}}{\beta'}\right|_{r_0=0}\right] = \lim_{r_{\rm max}\to\infty}\left[\frac{I_E^{\rm bulk}}{\beta} - \left.\frac{\beta}{\beta'}\frac{I_E^{\rm bulk}}{\beta}\right|_{r_0=0}\right], \tag{6.10}$$

where the ratio of the temperatures is given by

$$\frac{\beta}{\beta'} = f^{1/2}|_{r=r_{\rm max}} = 1 - \frac{1}{2}\left(\frac{r_0}{r_{\rm max}}\right)^{d-2} + \mathcal{O}(r_{\rm max}^{-d}). \tag{6.11}$$

We note that this expression is meaningful since there is no dependence on the angles to order $\mathcal{O}(r_{\rm max}^{-d})$. Substituting (6.9) and (6.11) in (6.10) we then find after taking the limit the result

$$F_{\rm bulk} = -\frac{V_p V(S^{d-1})}{16\pi G}\frac{(d-2)^2}{2(D-2)}r_0^{d-2}. \tag{6.12}$$

To find the boundary contribution we note that there are two boundaries, at $r = r_H$ and $r = r_{\rm max}$ respectively. The boundary action (6.3) then gives[20]

$$\begin{aligned}\frac{I_E^{\rm bd}}{\beta} &= -\frac{1}{\beta}\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{D-1}x\left(\partial_r(\sqrt{g}\sqrt{g^{rr}})\right)\sqrt{g^{rr}} \\ &= \frac{V_p V(S^{d-1})}{16\pi G}\left[\left(\frac{(p+1)(d-2)}{D-2} - 2(d-1)\right)(r_{\rm max}^{d-2} - r_0^{d-2}) - \right. \\ &\qquad \left. - (d-2)r_0^{d-2}\right]\left(1 + \mathcal{O}(r_{\rm max}^{-2})\right), \end{aligned} \tag{6.13}$$

[20] We thank J. Correia for useful discussions about this computation.

where we remark that only the boundary at $r = r_{\max}$ contributes. We note that here the r_0-dependent terms are either written explicitly or are of order $\mathcal{O}(r_{\max}^{-2})$. From (6.13) and (6.11) one then obtains the regularized boundary contribution to the free energy

$$F_{\rm bd} = \lim_{r_{\max}\to\infty} \left[\frac{I_E^{\rm bd}}{\beta} - \frac{\beta}{\beta'} \frac{I_E^{\rm bd}}{\beta}\bigg|_{r_0=0} \right] = -\frac{V_p V(S^{d-1})}{16\pi G} \left[\frac{(p+1)(d-2)}{2(D-2)} - 1 \right] r_0^{d-2} \,, \tag{6.14}$$

which we note vanishes for non-dilatonic branes, as seen using (2.2). Adding the two free energy contributions (6.12) and (6.14) we get

$$F = F_{\rm bulk} + F_{\rm bd} = -\frac{V_p V(S^{d-1})}{16\pi G} \frac{d-4}{2} r_0^{d-2} \,, \tag{6.15}$$

that precisely reproduces the thermodynamically computed Gibbs free energy (3.11). This fact will be used implicitly when we calculate string corrections to the free energy of the spinning D3-brane in section 6.2.

6.2 Corrections from higher derivative terms

In this section we test the conjecture that there exists smooth interpolation functions between strong and weak 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ for the D-branes, as discussed in section 5.1. The idea is to compute the correction to the free energy from the $l_s^6 R^4$ term in type-II string theory since this gives us the first correction in $1/\lambda$.

We restrict ourselves to the case of the D3-brane, since the constant dilaton for the non-corrected solution makes the computation considerably easier. The extremal non-rotating D3-brane has the geometry $AdS_5 \times S^5$ in the near-horizon limit and the dual field theory is the $N = 4$ $D = 4$ SYM [7]. For the spinning D3-brane the dual field theory is $N = 4$ $D = 4$ SYM at finite temperature with the R-voltage turned on. The free energy of the spinning D3-brane in the strong and weakly coupled limits has previously been discussed in [46, 24, 50].

We furthermore restrict ourselves to one non-zero angular momentum only but the methods we use can easily be extended to more angular momenta. To simplify the computations, we work in the limit $\omega \ll \pi$ with $\omega = \Omega/T$. This corresponds to the limit $l \ll r_0$. We develop all series in ω to order ω^4, with the next corrections coming from a ω^6 term. With this, our results are accurate for $\omega < 1$ up to about 1%.

Thus, we will test the interpolation between the free energy

$$F_{\lambda=0}(T,\Omega) = -N^2 V_3 T^4 \left(\frac{\pi^2}{6} + \frac{1}{4}\omega^2 - \frac{1}{32\pi^2}\omega^4 \right) \tag{6.16}$$

for weak coupling, obtained from (5.6c), and the free energy

$$F_{\lambda=\infty}(T,\Omega) = -N^2 V_3 T^4 \left(\frac{\pi^2}{8} + \frac{1}{8}\omega^2 + \frac{1}{16\pi^2}\omega^4 + \mathcal{O}(\omega^6) \right) \tag{6.17}$$

for strong coupling, obtained from (3.21). In this case we can write the interpolation conjecture (5.8) as

$$F_\lambda(T,\Omega) = f(\lambda,\omega) F_{\lambda=0}(T,\Omega)\,, \tag{6.18}$$

where $f = f(\lambda,\omega)$ is the interpolation function. To zeroth order in $1/\lambda$ we thus have

$$f(\lambda=\infty,\omega) = \frac{3}{4} - \frac{3}{8\pi^2}\omega^2 + \frac{69}{64\pi^4}\omega^4 + \mathcal{O}(\omega^6)\,. \tag{6.19}$$

Comparing (6.16) and (6.17) we see that we should expect $f(\lambda,\omega)$ to be smaller than one, and we also expect it to be decreasing with λ, for fixed $\omega < 1$, since

$$-F_{\lambda=\infty} < -F_{\lambda=0} \text{ for } \omega < 1 \tag{6.20}$$

i.e. the absolute value of the free energy for $\lambda = \infty$ is less than the one for $\lambda = 0$ for $\omega < 1$. In ref. [52] the interpolation function $f(\lambda,\omega)$ was studied for $\omega = 0$, and it was found that

$$f(\lambda,0) = \frac{3}{4} + \frac{45}{32}\zeta(3)(2\lambda)^{-3/2} + \cdots \tag{6.21}$$

by computation of the correction from the $l_s^6 R^4$ term in type-IIB string theory. The computation (6.21) clearly supports the conjecture that there is a monotonous smooth interpolation function, since the $\lambda^{-3/2}$ correction is positive.[21]

As previously stated, the higher derivative correction term[22] in the supergravity action for type-IIB string theory that we want to consider is the $l_s^6 R^4$ term. In the euclidean case, the term is given in the Einstein frame by

$$\delta I_E = -\frac{1}{16\pi G}\int d^{10}x\sqrt{g}\,\gamma e^{-\frac{3}{2}\phi}W\,, \tag{6.22}$$

with $\gamma = \frac{1}{8}\zeta(3) l_s^6$ and

$$W = C^{\mu_1\mu_2\mu_3\mu_4} C_{\nu_1\mu_2\mu_3\nu_4} C_{\mu_1}{}^{\nu_2\nu_3\nu_1} C^{\nu_4}{}_{\nu_2\nu_3\mu_4} + \frac{1}{2} C^{\mu_1\mu_2\mu_3\mu_4} C_{\nu_1\nu_4\mu_3\mu_4} C_{\mu_1}{}^{\nu_2\nu_3\nu_1} C^{\nu_4}{}_{\nu_2\nu_3\mu_2}\,, \tag{6.23}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. In the near-horizon limit (3.1a) with $\ell = l_s \to 0$ we have the same term (6.22) in terms of the rescaled quantities, but with γ rescaled to

$$\gamma = \frac{1}{8}\zeta(3)\,. \tag{6.24}$$

[21] On the weak coupling side, a two loop calculation [62] has shown that the leading correction in λ is negative, giving further evidence for the interpolation conjecture. In [63] further corrections in λ from the weak coupling side are considered and also found to support the interpolation conjecture.

[22] For dual field theories with a smaller amount of supersymmetry, refs. [64, 65] consider analogous higher derivative corrections to obtain the modification of the thermodynamics.

From (3.20) we furthermore have the relations

$$h^4 = 2\lambda\,, \qquad \frac{V(S^5)h^8}{16\pi G} = \frac{N^2}{8\pi^2}\,. \tag{6.25}$$

String theory admits two different kinds of expansions, the loop expansion in g_s and the derivative expansion in $\alpha' = l_s^2$. In particular, for the type-IIB R^4 term there is also a one-loop term of the form $g_s^2 l_s^6 R^4$, and an infinite sum of D-instanton corrections. Through the AdS/CFT correspondence this translates into a $1/N$ and $1/\lambda$ expansion (see e.g. [66] which also discusses the instanton corrections). The $l_s^6 R^4$ tree-level term becomes then a $\lambda^{-3/2}R^4$ term, while the $g_s^2 l_s^6 R^4$ one loop term becomes an $N^{-2}\lambda^{1/2}R^4$ term. The $N^{-2}\lambda^{1/2}R^4$ term is clearly not interesting for this computation, since we want to keep N fixed and large. It is also subleading since we keep g_s small in the limit we consider.

We now compute the $1/\lambda$ correction to the free energy (6.17) by inserting the non-corrected Wick rotated solution (C.3a) for the euclidean spinning D3-brane in the near-horizon limit, into the higher derivative term (6.22) as the background geometry. Substituting the solution we find for the first three terms[23] in a weak angular momentum expansion

$$\begin{aligned} W = \frac{180}{h^8}\left(\frac{r_0}{r}\right)^{16}\Bigg[1 + \frac{2}{3}\left[10\cos^2\theta + (4 - 5\cos^2\theta)\left(\frac{r}{r_0}\right)^4\right]\left(\frac{\tilde{l}}{r}\right)^2 + \\ + \frac{1}{120}\Bigg[3617\cos^4\theta + (-3032\cos^2\theta + 2288)\cos^2\theta\left(\frac{r}{r_0}\right)^4 + \\ + (512 - 904\cos^2\theta + 644\cos^4\theta)\left(\frac{r}{r_0}\right)^8\Bigg]\left(\frac{\tilde{l}}{r}\right)^4 + \cdots\Bigg]\,, \end{aligned} \tag{6.26}$$

while the volume element is

$$\sqrt{g} = h^2 r^3\cos^3\theta\sin\theta\cos\psi_1\sin\psi_1\left[1 - \frac{1}{2}\cos^2\theta\left(\frac{\tilde{l}}{r}\right)^2 - \frac{1}{8}\cos^4\theta\left(\frac{\tilde{l}}{r}\right)^4 + \cdots\right] \tag{6.27}$$

Substituting this in (6.22), integrating over the angles, the world-volume, the euclidean time τ from 0 to β, and r from r_H to infinity, we arrive at

$$\delta F = \frac{\delta I_E}{\beta} = -\frac{V_3 V(S^5)}{16\pi G}\frac{\gamma}{h^6}r_0^4\left[15 + \frac{111}{7}\left(\frac{\tilde{l}}{r_0}\right)^2 + \frac{5885}{96}\left(\frac{\tilde{l}}{r_0}\right)^4 + \cdots\right]. \tag{6.28}$$

[23]We have obtained the exact result but refrain from giving this rather lengthy expression.

Here we have also used the expansion for the horizon radius

$$r_H = r_0 \left[1 + \frac{1}{4}\left(\frac{\tilde{l}}{r_0}\right)^2 + \frac{1}{32}\left(\frac{\tilde{l}}{r_0}\right)^4 + \cdots \right], \tag{6.29}$$

which follows from (D.3) with one angular momentum turned on only and the replacement $l = i\tilde{l}$.

Finally, we write the result (6.28) in terms of the thermodynamic parameters T, Ω using (D.7), (D.5b) which imply,

$$r_0^4 = (Th^2)^4 \left[1 + \frac{1}{\pi^2}\omega^2 + \frac{1}{2\pi^4}\omega^4 + \mathcal{O}(\omega^6) \right] \tag{6.30a}$$

$$-\left(\frac{\tilde{l}}{r_0}\right)^2 = \frac{1}{\pi^2}\omega^2 + \frac{1}{2\pi^4}\omega^4 + \mathcal{O}(\omega^6) \,. \tag{6.30b}$$

We also use the relations (6.25) that enable to transform to the field theory parameters. Then we obtain the final form of the correction

$$\delta F = -\frac{\zeta(3)\pi^2}{64}(2\lambda)^{-3/2} N^2 V_3 T^4 \left[15 - \frac{6}{7\pi^2}\omega^2 + \frac{28151}{672\pi^4}\omega^4 + \mathcal{O}(\omega^6) \right]. \tag{6.31}$$

This gives the interpolation function

$$f(\lambda,\omega) = \frac{3}{4} - \frac{3}{8\pi^2}\omega^2 + \frac{69}{64\pi^4}\omega^4 + \frac{\zeta(3)}{64}(2\lambda)^{-3/2}\left(90 - \frac{981}{7\pi^2}\omega^2 + \frac{7655}{16\pi^4}\omega^4 \right) + \cdots \tag{6.32}$$

which includes (6.19) and (6.21) as special cases. Because of (6.20) we expect the correction term in (6.32) to be positive for $\omega < 1$ and this is indeed the case. Since the corrections away from $\lambda = \infty$ behave as expected, we consider this as further evidence for the interpolation conjecture (6.18) in the case of spinning D3-branes,

It is not a priori apparent that the method used to compute (6.32) gives the full result to first order in γ, since this semi-classical method does not consider induced perturbations of the geometry. However, we note that for the non-rotating D3-brane case, the perturbed metric induced by the correction term (6.22) was shown [52, 53] to yield the same correction to the free energy as the semiclassical approximation in which the correction is evaluated for the original unperturbed metric.[24] Thus, it seems that the thermodynamics somehow disregards perturbations of the geometry. We expect this to hold also for non-dilatonic spinning branes. It would be interesting, though technically difficult, to find the actual perturbed metric for the spinning D3-brane and determine whether the result is still given by (6.32). Another interesting check on the interpolation function (6.32) would be to compute the $1/\lambda$ correction to the boundary of stability. From the values in table 5 we would expect this correction to be positive.

[24]This feature has been shown to persist as well for the corrections to the near-horizon M2 and M5-brane backgrounds originating from the R^4 term in M-theory [67].

Acknowledgments

We thank J. Ambjørn, E. Cheung, P. Damgaard, P. Di Vecchia, E. Kiritsis, F. Larsen, R. Marotta, J. L. Petersen, B. Pioline, K. Savvidis, K. Sfetsos and R. Szabo for useful discussions and correspondence, and we thank J. Correia for useful discussions and early participation on the subjects of section 2. NO thanks the CERN Theory Division for support and hospitality during completion of this work. This work is supported in part by TMR network ERBFMRXCT96-0045.

A. Thermodynamics of black p-branes in the near-horizon limit

In this appendix we consider the thermodynamics, and in particular the energy above extremality in the near-horizon limit of a general non-rotating p-brane. We also derive the Smarr formula and check that it is fulfilled. This means that the first law of thermodynamics is obeyed in the near-horizon limit, and that there are no extra thermodynamic parameters related to the charge. We begin by giving a short review of the classification of p-branes that preserve a certain fraction of the supersymmetry (in the extremal limit).

A black p-brane solution of the action (2.1) is characterized by a particular value of a. If we define

$$b \equiv \frac{2(D-2)}{(p+1)(d-2)+\frac{1}{2}a^2(D-2)} \tag{A.1}$$

then the non-rotating black p-brane background takes the form

$$\begin{aligned} ds^2 &= H^{-\frac{d-2}{D-2}b}\left(-f\,dt^2+\sum_{i=1}^{p}(dy^i)^2\right)+H^{\frac{p+1}{D-2}b}\big(f^{-1}dr^2+r^2d\Omega_{d-1}^2\big)\,, && \text{(A.2a)}\\ e^{\phi} &= H^{\frac{a}{2}b}\,, && \text{(A.2b)}\\ A_{p+1} &= (-1)^p\sqrt{b}\coth\alpha\big(H^{-1}-1\big)\,dt\wedge dy^1\wedge dy^2\wedge\cdots\wedge dy^p\,, && \text{(A.2c)} \end{aligned}$$

with

$$H = 1+\frac{r_0^{d-2}\sinh^2\alpha}{r^{d-2}}\,,\qquad f = 1-\frac{r_0^{d-2}}{r^{d-2}}\,. \tag{A.3}$$

This p-brane solution is a $1/2^b$-BPS state [12] for $r_0 = 0$, so that the spinning solutions discussed in the text correspond to $b = 1$. Table 6 lists the most common branes together with the corresponding values of D, a and b (a more extensive list with other values of b can be found in ref. [12]).

The thermodynamic quantities of the background (A.2) are

$$M = \frac{V_p V(S^{d-1})}{16\pi G} r_0^{d-2} \Big[d - 1 + b(d-2)\sinh^2\alpha \Big] \tag{A.4a}$$

$$T = \frac{d-2}{4\pi r_0} (\cosh\alpha)^{-b}\,, \qquad S = \frac{V_p V(S^{d-1})}{4G} r_0^{d-1} (\cosh\alpha)^b \tag{A.4b}$$

$$\mu = \tanh\alpha\,, \qquad Q = \frac{V_p V(S^{d-1})}{16\pi G} b(d-2) r_0^{d-2} \cosh\alpha \sinh\alpha \tag{A.4c}$$

satisfying the Smarr formula

$$(d-2)M = (d-1)TS + (d-2)\mu Q \tag{A.5}$$

and the first law of thermodynamics

$$dM = TdS + \mu dQ\,, \quad M = M(S,Q)\,. \tag{A.6}$$

Brane	Theory	D	a	b
M2	M	11	0	1
M5	M	11	0	1
Dp	string	10	$(3-p)/2$	1
NS1	string	10	-1	1
NS5	string	10	1	1
dp	little string	6	$1-p$	2

Table 6: The characteristic numbers a and b for branes with b equal to 1 or 2.

The energy above extremality is

$$E = M - Q = \frac{V_p V(S^{d-1})}{16\pi G} r_0^{d-2} \Big[d - 1 + b(d-2)(\sinh^2\alpha - \cosh\alpha\sinh\alpha) \Big]\,. \tag{A.7}$$

The near-horizon limit is defined via the rescaling

$$r = \frac{r_{\rm old}}{\ell^2}\,, \qquad r_0 = \frac{(r_0)_{\rm old}}{\ell^2}\,, \qquad h^{d-2} = \frac{h_{\rm old}^{d-2}}{\ell^{2d-4-\frac{4}{b}}} \tag{A.8a}$$

$$ds^2 = \frac{(ds^2)_{\rm old}}{\ell^{4(d-2)b/(D-2)}}\,, \qquad e^\phi = \ell^{2a} e^{\phi_{\rm old}}\,, \qquad A = \frac{A_{\rm old}}{\ell^4}\,, \qquad G = \frac{G_{\rm old}}{\ell^{2(d-2)}} \tag{A.8b}$$

and taking $\ell \to 0$, keeping the old quantities fixed. Note that the near-horizon limit depends on the fraction of supersymmetries that is preserved and that we recover (3.1a) for $b = 1$. Using this limit in (A.4b) and (A.7) we obtain

$$T = \frac{d-2}{4\pi r_0} \left(\frac{r_0}{h}\right)^{\frac{d-2}{2}b}\,, \qquad S = \frac{V_p V(S^{d-1})}{4G} r_0^{d-1} \left(\frac{h}{r_0}\right)^{\frac{d-2}{2}b}\,, \tag{A.9a}$$

$$E = \frac{V_p V(S^{d-1})}{16\pi G} \Big[d - 1 - \frac{b}{2}(d-2) \Big] r_0^{d-2}\,. \tag{A.9b}$$

To derive the Smarr formula we consider the canonical rescaling

$$h \to h\,, \qquad r_0 \to \lambda r_0 \tag{A.10}$$

under which we have the transformation

$$E \to \lambda^{d-2}\,, \qquad S \to \lambda^{(1-\frac{1}{2}b)d+b-1} S \tag{A.11}$$

This gives the Smarr formula

$$(d-2)E = \Big(d-1-\frac{b}{2}(d-2)\Big)TS \tag{A.12}$$

corresponding to the first law of thermodynamics

$$dE = TdS\,, \qquad E = E(S)\,. \tag{A.13}$$

The Smarr formula (A.12) is indeed satisfied with (A.9). We note that (A.12) is qualitatively different from the asymptotically-flat black brane Smarr formula (A.5) since it exhibits a dependence on the amount of unbroken supersymmetry that the brane has in the extremal limit.

The free energy is given by

$$F = E - TS = -\frac{\frac{b}{2}(d-2)-1}{d-1-\frac{b}{2}(d-2)}E = -\frac{V_p V(S^{d-1})}{16\pi G}\left[\frac{b}{2}(d-2)-1\right]r_0^{d-2}\,. \tag{A.14}$$

In particular, for $b=1$ we have

$$E = \frac{V_p V(S^{d-1})}{16\pi G}\frac{d}{2}r_0^{d-2}\,, \qquad F = -\frac{d-4}{d}E = -\frac{V_p V(S^{d-1})}{16\pi G}\frac{d-4}{2}r_0^{d-2}\,, \tag{A.15}$$

while for $b=2$ the result reads

$$E = \frac{V_p V(S^{d-1})}{16\pi G}r_0^{d-2}\,, \qquad F = -(d-3)E = -\frac{V_p V(S^{d-1})}{16\pi G}(d-3)r_0^{d-2}\,. \tag{A.16}$$

B. Spheroidal coordinates

In this appendix we define the spheroidal coordinates for a d-dimensional euclidean space with cartesian coordinates $x^a, a = 1, \ldots, d$. We define the metric

$$(ds_d)^2 = \sum_{a=1}^{d}(dx^a)^2 \tag{B.1}$$

and treat the cases d even and odd separately.

The case $d = 2n$. The spheroidal coordinates are the "radius" r and the angles $\theta, \psi_1, \ldots, \psi_{n-2},, \phi_1, \ldots, \phi_n$. Define the quantities

$$\begin{aligned}&\mu_1 = \sin\theta, \quad \mu_2 = \cos\theta\sin\psi_1, \quad \mu_3 = \cos\theta\cos\psi_1\sin\psi_2\,, \ldots \\ &\ldots, \mu_{n-1} = \cos\theta\cos\psi_1\cdots\cos\psi_{n-3}\sin\psi_{n-2}, \quad \mu_n = \cos\theta\cos\psi_1\cdots\cos\psi_{n-2}\,,\end{aligned} \tag{B.2}$$

which satisfy

$$\sum_{i=1}^{n}\mu_i^2 = 1\,. \tag{B.3}$$

The spheroidal coordinates are then defined by

$$x^{2i-1} = \sqrt{r^2 + l_i^2}\mu_i \cos\phi_i\,, \qquad x^{2i} = \sqrt{r^2 + l_i^2}\mu_i \sin\phi_i\,, \qquad i = 1,\dots,n\,. \tag{B.4}$$

The coordinates $\phi_1,\dots,\phi_n$ are the rotation angles and $l_1,\dots,l_n$ correspond to the angular momenta in these angles. We have

$$\sum_{a=1}^{d}(x^a)^2 = r^2 + \sum_{i=1}^{n} l_i^2\mu_i^2 \tag{B.5}$$

and the ranges of the angles are given by

$$0 \le \theta, \psi_1,\dots,\psi_{n-2} \le \frac{\pi}{2}\,, \qquad 0 \le \phi_1,\dots,\phi_n \le 2\pi\,. \tag{B.6}$$

The case $d = 2n+1$. The spheroidal coordinates are the “radius” r and the angles $\theta, \psi_1,\dots,\psi_{n-1}$, $\phi_1,\dots,\phi_n$. Define the quantities

$$\begin{aligned} &\mu_1 = \sin\theta, \quad \mu_2 = \cos\theta\sin\psi_1, \quad \mu_3 = \cos\theta\cos\psi_1\sin\psi_2\,,\dots \\ &\dots,\mu_n = \cos\theta\cos\psi_1\cdots\cos\psi_{n-2}\sin\psi_{n-1}, \quad \mu_{n+1} = \cos\theta\cos\psi_1\cdots\cos\psi_{n-1}\,, \end{aligned} \tag{B.7}$$

which satisfy

$$\sum_{i=1}^{n+1}\mu_i^2 = 1\,. \tag{B.8}$$

The spheroidal coordinates are then defined by

$$\begin{aligned} x^{2i-1} &= \sqrt{r^2 + l_i^2}\mu_i \cos\phi_i\,, \qquad x^{2i} = \sqrt{r^2 + l_i^2}\mu_i \sin\phi_i\,, \qquad i = 1,\dots,n \\ x^d &= r\mu_{n+1}\,. \end{aligned} \tag{B.9}$$

The coordinates $\phi_1,\dots,\phi_n$ are the rotation angles and $l_1,\dots,l_n$ corresponds to the angular momenta in these angles. In this case, we have

$$\sum_{a=1}^{d}(x^a)^2 = r^2 + \sum_{i=1}^{n} l_i^2\mu_i^2\,. \tag{B.10}$$

The ranges of the angles are, for $d \ge 5$, given by

$$0 \le \theta, \psi_1,\dots,\psi_{n-2} \le \frac{\pi}{2}\,, \qquad 0 \le \psi_{n-1} \le \pi\,, \qquad 0 \le \phi_1,\dots,\phi_n \le 2\pi \tag{B.11}$$

and for $d = 3$ we have

$$0 \le \theta \le \pi\,, \qquad 0 \le \phi_1 \le 2\pi\,. \tag{B.12}$$

The spheroidal metric. The metric in spheroidal coordinates takes the form

$$\sum_{a=1}^{d}(dx^a)^2 = K_d dr^2 + \Lambda_{\alpha\beta} d\eta^\alpha d\eta^\beta \,, \tag{B.13}$$

where η^α denote the set of angular coordinates. For the radial coordinate the metric component takes the form

$$g_{rr} = K_d(r,\theta,\psi_1,\ldots,\psi_{d-n-2}) \equiv \begin{cases} \sum_{i=1}^{n} \mu_i^2 \left(1+\frac{l_i^2}{r^2}\right)^{-1} , & d=2n \\ \sum_{i=1}^{n} \mu_i^2 \left(1+\frac{l_i^2}{r^2}\right)^{-1} + \mu_{n+1}^2 \,, & d=2n+1 \end{cases} \tag{B.14}$$

and the general form of the remaining non-zero components is

$$\begin{aligned}
g_{\theta\theta} &= r^2 + l_1^2\cos^2\theta + \tan^2\theta\left(\mu_2^2 l_2^2 + \cdots + \mu_n^2 l_n^2\right) \\
g_{\psi_1\psi_1} &= \cos^2\theta\left(r^2 + l_2^2\cos^2\psi_1\right) + \tan^2\psi_1\left(\mu_3^2 l_3^2 + \cdots + \mu_n^2 l_n^2\right) \\
g_{\psi_2\psi_2} &= \cos^2\theta\cos^2\psi_1\left(r^2 + l_3^2\cos^2\psi_2\right) + \tan^2\psi_2\left(\mu_4^2 l_4^2 + \cdots + \mu_n^2 l_n^2\right) \\
g_{\psi_3\psi_3} &= \cos^2\theta\cos^2\psi_1\cos^2\psi_2\left(r^2 + l_4^2\cos^2\psi_3\right) + \tan^2\psi_3\left(\mu_5^2 l_5^2 + \cdots + \mu_n^2 l_n^2\right) \\
g_{\psi_{n-2}\psi_{n-2}} &= \cos^2\theta\cos^2\psi_1\cdots\cos^2\psi_{n-3}\left(r^2 + l_{n-1}^2\cos^2\psi_{n-2} + l_n^2\sin^2\psi_{n-2}\right) \\
g_{\theta\psi_1} &= -\tan\theta\cot\psi_1\mu_2^2 l_2^2 + \tan\theta\tan\psi_1\left(\mu_3^2 l_3^2 + \cdots + \mu_n^2 l_n^2\right) \\
g_{\theta\psi_2} &= -\tan\theta\cot\psi_2\mu_3^2 l_3^2 + \tan\theta\tan\psi_2\left(\mu_4^2 l_4^2 + \cdots + \mu_n^2 l_n^2\right) \\
g_{\psi_1\psi_2} &= -\tan\psi_1\cot\psi_2\mu_3^2 l_3^2 + \tan\psi_1\tan\psi_2\left(\mu_4^2 l_4^2 + \cdots + \mu_n^2 l_n^2\right) \\
g_{\phi_i\phi_i} &= \mu_i^2(r^2 + l_i^2)\,, \qquad i = 1,\ldots,n
\end{aligned}$$

for both $d = 2n$ and $d = 2n+1$. As an aid to the reader we list below the angles and explicit expressions for the spheroidal metric when $3 \le d \le 9$.

$$d=3: \quad \theta,\phi_1 \tag{B.15a}$$

$$(ds_3)^2 = K_3 dr^2 + \left(r^2 + l_1^2\cos^2\theta\right)d\theta^2 + \sin^2\theta\left(r^2 + l_1^2\right)d\phi_1^2\,;$$

$$d=4: \quad \theta,\phi_1,\phi_2 \tag{B.15b}$$

$$\begin{aligned}
(ds_4)^2 = {}& K_4 dr^2 + \left(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta\right)d\theta^2 + \sin^2\theta\left(r^2 + l_1^2\right)d\phi_1^2 + \\
&+ \cos^2\theta\left(r^2 + l_2^2\right)d\phi_2^2\,;
\end{aligned}$$

$$d=5: \quad \theta, \psi_1, \phi_1, \phi_2 \tag{B.15c}$$

$$\begin{aligned}(ds_5)^2 =&\ K_5 dr^2 + \Big(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta\sin^2\psi_1\Big)d\theta^2 + \\ &+ \cos^2\theta\Big(r^2 + l_2^2\cos^2\psi_1\Big)d\psi_1^2 + -2l_2^2\cos\theta\sin\theta\cos\psi_1\sin\psi_1 d\theta d\psi_1 + \\ &+ \sin^2\theta\Big(r^2+l_1^2\Big)d\phi_1^2 + \cos^2\theta\sin^2\psi_1\Big(r^2+l_2^2\Big)d\phi_2^2\,;\end{aligned}$$

$$d=6: \quad \theta, \psi_1, \phi_1, \phi_2, \phi_3 \tag{B.15d}$$

$$\begin{aligned}(ds_6)^2 =&\ K_6 dr^2 + \Big(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta\sin^2\psi_1 + l_3^2\sin^2\theta\cos^2\psi_1\Big)d\theta^2 + \\ &+ \cos^2\theta\Big(r^2 + l_2^2\cos^2\psi_1 + l_3^2\sin^2\psi_1\Big)d\psi_1^2 + \\ &+ 2\cos\theta\sin\theta\cos\psi_1\sin\psi_1\Big(-l_2^2+l_3^2\Big)d\theta d\psi_1 + \sin^2\theta\Big(r^2+l_1^2\Big)d\phi_1^2 + \\ &+ \cos^2\theta\sin^2\psi_1\Big(r^2+l_2^2\Big)d\phi_2^2 + \cos^2\theta\cos^2\psi_1\Big(r^2+l_3^2\Big)d\phi_3^2\,;\end{aligned}$$

$$d=7: \quad \theta, \psi_1, \psi_2, \phi_1, \phi_2, \phi_3 \tag{B.15e}$$

$$\begin{aligned}(ds_7)^2 =&\ K_7 dr^2 + \Big(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta\sin^2\psi_1 + l_3^2\sin^2\theta\cos^2\psi_1\sin^2\psi_2\Big)d\theta^2 + \\ &+ \cos^2\theta\Big(r^2 + l_2^2\cos^2\psi_1 + l_3^2\sin^2\theta\sin^2\psi_2\Big)d\psi_1^2 + \\ &+ \cos^2\theta\cos^2\psi_1\Big(r^2 + l_3^2\cos^2\psi_2\Big)d\psi_2^2 + \\ &+ 2\cos\theta\sin\theta\cos\psi_1\sin\psi_1\Big(-l_2^2 + l_3^2\sin^2\psi_2\Big)d\theta d\psi_1 - \\ &- 2\cos\theta\sin\theta\cos^2\psi_1\cos\psi_2\sin\psi_2 l_3^2 d\theta d\psi_2 \\ &-2l_3^2\cos\psi_1\sin\psi_1\cos^2\psi_1\cos\psi_2\sin\psi_2 d\psi_1 d\psi_2 + \sin^2\theta\Big(r^2+l_1^2\Big) + d\phi_1^2 \\ &+ \cos^2\theta\sin^2\psi_1\Big(r^2+l_2^2\Big)d\phi_2^2 + \cos^2\theta\cos^2\psi_1\sin^2\psi_2\Big(r^2+l_3^2\Big)d\phi_3^2\end{aligned}$$

$$d=8: \quad \theta, \psi_1, \psi_2, \phi_1, \phi_2, \phi_3, \phi_4 \tag{B.15f}$$

$$\begin{aligned}(ds_8)^2 =&\ K_8 dr^2 + \Big(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta\sin^2\psi_1 + l_3^2\sin^2\theta\cos^2\psi_1\sin^2\psi_2 + \\ &+ l_4^2\sin^2\theta\cos^2\psi_1\cos^2\psi_2\Big)d\theta^2 \\ &+ \cos^2\theta\Big(r^2 + l_2^2\cos^2\psi_1 + l_3^2\sin^2\psi_1\sin^2\psi_2 + l_4^2\sin^2\psi_1\cos^2\psi_2\Big)d\psi_1^2 + \\ &+ \cos^2\theta\cos^2\psi_1\Big(r^2 + l_3^2\cos^2\psi_2 + l_4^2\sin^2\psi_2\Big)d\psi_2^2 + \\ &+ 2\cos\theta\sin\theta\cos\psi_1\sin\psi_1\Big(-l_2^2 + l_3^2\sin^2\psi_2 + l_4^2\cos^2\psi_2\Big)d\theta d\psi_1 + \\ &+ 2\cos\theta\sin\theta\cos^2\psi_1\cos\psi_2\sin\psi_2\Big(-l_3^2+l_4^2\Big)d\theta d\psi_2 + \\ &+ 2\cos^2\theta\cos\psi_1\sin\psi_1\cos\psi_2\sin\psi_2\Big(-l_3^2+l_4^2\Big)d\psi_1 d\psi_2 +\end{aligned}$$

$$+\ \sin^2\theta\Big(r^2+l_1^2\Big)d\phi_1^2+\cos^2\theta\sin^2\psi_1\Big(r^2+l_2^2\Big)d\phi_2^2+$$
$$+\ \cos^2\theta\cos^2\psi_1\sin^2\psi_2\Big(r^2+l_3^2\Big)d\phi_3^2+$$
$$+\ \cos^2\theta\cos^2\psi_1\cos^2\psi_2\Big(r^2+l_4^2\Big)d\phi_4^2\,;$$

$$d=9:\quad \theta,\psi_1,\psi_2,\psi_3,\phi_1,\phi_2,\phi_3,\phi_4 \tag{B.15g}$$

$$\begin{aligned}
(ds_9)^2 =\ & K_9 dr^2+\Big(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta\sin^2\psi_1+l_3^2\sin^2\theta\cos^2\psi_1\sin^2\psi_2+\\
&+l_4^2\sin^2\theta\cos^2\psi_1\cos^2\psi_2\sin^2\psi_3\Big)d\theta^2+\\
&+\ \cos^2\theta\Big(r^2+l_2^2\cos^2\psi_1+l_3^2\sin^2\psi_1\sin^2\psi_2+l_4^2\sin^2\psi_1\cos^2\psi_2\sin^2\psi_3\Big)d\psi_1^2+\\
&+\ \cos^2\theta\cos^2\psi_1\Big(r^2+l_3^2\cos^2\psi_2+l_4^2\sin^2\psi_2\sin^2\psi_3\Big)d\psi_2^2+\\
&+\ \cos^2\theta\cos^2\psi_1\cos^2\psi_2\Big(r^2+l_4^2\cos^2\psi_3\Big)d\psi_3^2+\\
&+2\cos\theta\sin\theta\cos\psi_1\sin\psi_1\Big(-l_2^2+l_3^2\sin^2\psi_2+l_4^2\cos^2\psi_2\sin^2\psi_3\Big)d\theta d\psi_1+\\
&+2\cos\theta\sin\theta\cos^2\psi_1\cos\psi_2\sin\psi_2\Big(-l_3^2+l_4^2\sin^2\psi_3\Big)d\theta d\psi_2-\\
&-2l_4^2\cos\theta\sin\theta\cos^2\psi_1\cos^2\psi_2\cos\psi_3\sin\psi_3 d\theta d\psi_3+\\
&+2\cos^2\theta\cos\psi_1\sin\psi_1\cos\psi_2\sin\psi_2\Big(-l_3^2+l_4^2\sin^2\psi_3\Big)d\psi_1 d\psi_2-\\
&-2l_4^2\cos^2\theta\cos^2\psi_1\cos\psi_2\sin\psi_2\cos\psi_3\sin\psi_3 d\psi_2 d\psi_3+\sin^2\theta\Big(r^2+l_1^2\Big)d\phi_1^2+\\
&+\ \cos^2\theta\sin^2\psi_1\Big(r^2+l_2^2\Big)d\phi_2^2+\cos^2\theta\cos^2\psi_1\sin^2\psi_2\Big(r^2+l_3^2\Big)d\phi_3^2+\\
&+\ \cos^2\theta\cos^2\psi_1\cos^2\psi_2\sin^2\psi_3\Big(r^2+l_4^2\Big)d\phi_4^2\,.
\end{aligned}$$

Spheroidal metric with one angular momentum. As an aid to the reader we also give the explicit expressions for the spheroidal metric when only one angular momentum $l_1=l$ is non-zero,

$$(ds_3)^2 = \Big(1-\frac{l^2\sin^2\theta}{l^2+r^2}\Big)dr^2+\Big(r^2+l^2\cos^2\theta\Big)d\theta^2+\sin^2\theta\Big(r^2+l^2\Big)d\phi_1^2\,, \tag{B.16a}$$

$$\begin{aligned}(ds_4)^2 =\ & \Big(1-\frac{l^2\sin^2\theta}{l^2+r^2}\Big)dr^2+\Big(r^2+l^2\cos^2\theta\Big)d\theta^2+\\
&+\sin^2\theta\Big(r^2+l^2\Big)d\phi_1^2+r^2\cos^2\theta d\phi_2^2\,,\end{aligned} \tag{B.16b}$$

$$\begin{aligned}(ds_5)^2 =\ & \Big(1-\frac{l^2\sin^2\theta}{l^2+r^2}\Big)dr^2+\Big(r^2+l^2\cos^2\theta\Big)d\theta^2+r^2\cos^2\theta d\psi_1^2+\\
&+\sin^2\theta\Big(r^2+l^2\Big)d\phi_1^2+r^2\cos^2\theta\sin^2\psi_1 d\phi_2^2\,,\end{aligned} \tag{B.16c}$$

$$(ds_6)^2 = \Big(1 - \frac{l^2 \sin^2\theta}{l^2+r^2}\Big)dr^2 + \Big(r^2 + l^2\cos^2\theta\Big)d\theta^2 + r^2\cos^2\theta d\psi_1^2 + \\ + \sin^2\theta\Big(r^2+l^2\Big)d\phi_1^2 + r^2\cos^2\theta\sin^2\psi_1 d\phi_2^2 + r^2\cos^2\theta\cos^2\psi_1 d\phi_3^2\,, \quad \text{(B.16d)}$$

$$(ds_7)^2 = \Big(1 - \frac{l^2 \sin^2\theta}{l^2+r^2}\Big)dr^2 + \Big(r^2 + l^2\cos^2\theta\Big)d\theta^2 + r^2\cos^2\theta d\psi_1^2 + \\ + r^2\cos^2\theta\cos^2\psi_1 d\psi_2^2 + \sin^2\theta\Big(r^2+l^2\Big)d\phi_1^2 + r^2\cos^2\theta\sin^2\psi_1 d\phi_2^2 + \\ + r^2\cos^2\theta\cos^2\psi_1\sin^2\psi_2 d\phi_3^2\,, \quad \text{(B.16e)}$$

$$(ds_8)^2 = \Big(1 - \frac{l^2 \sin^2\theta}{l^2+r^2}\Big)dr^2 + \Big(r^2 + l^2\cos^2\theta\Big)d\theta^2 + r^2\cos^2\theta d\psi_1^2 + \\ + r^2\cos^2\theta\cos^2\psi_1 d\psi_2^2 + \sin^2\theta\Big(r^2+l^2\Big)d\phi_1^2 + r^2\cos^2\theta\sin^2\psi_1 d\phi_2^2 + \\ + r^2\cos^2\theta\cos^2\psi_1\sin^2\psi_2 d\phi_3^2 + r^2\cos^2\theta\cos^2\psi_1\cos^2\psi_2 d\phi_4^2\,, \quad \text{(B.16f)}$$

$$(ds_9)^2 = \Big(1 - \frac{l^2 \sin^2\theta}{l^2+r^2}\Big)dr^2 + \Big(r^2 + l^2\cos^2\theta\Big)d\theta^2 + r^2\cos^2\theta d\psi_1^2 + \\ + r^2\cos^2\theta\cos^2\psi_1 d\psi_2^2 + r^2\cos^2\theta\cos^2\psi_1\cos^2\psi_2 d\psi_3^2 + \\ + \sin^2\theta\Big(r^2+l^2\Big)d\phi_1^2 + r^2\cos^2\theta\sin^2\psi_1 d\phi_2^2 + \\ + r^2\cos^2\theta\cos^2\psi_1\sin^2\psi_2 d\phi_3^2 + r^2\cos^2\theta\cos^2\psi_1\cos^2\psi_2\sin^2\psi_3 d\phi_4^2\,. \quad \text{(B.16g)}$$

C. The euclidean near-horizon solution

In this appendix we give the euclidean version of the near-horizon solution (3.2) for one non-zero angular momentum. This comes into play in sections 6.1 and 6.2 when calculating the value of the action and of the corrected action. The euclidean solutions can simply be obtained by performing the Wick rotation

$$\tau = it\,, \qquad \tilde{l}_i = -il_i\,. \quad \text{(C.1)}$$

This induces the replacement $l_i^2 \to -\tilde{l}_i^2$ in the definitions of L_d of (2.7a) and K_d, $\Lambda_{\alpha\beta}$ of the spheroidal metric (2.6). In addition, we find that in the metric (3.2a) we have $-f dt^2 \to f d\tau^2$ as well as $l_i dt d\phi_i \to \tilde{l}_i d\tau d\phi_i$ in the off-diagonal terms. Finally, in the electric $(p+1)$-form potential we replace

$$\left(H^{-1}d\tau + \frac{r_0^{\frac{d-2}{2}}}{h^{\frac{d-2}{2}}}\sum_{i=1}^{n} l_i\mu_i^2 d\phi_i\right) \to -i\left(H^{-1}d\tau - \frac{r_0^{\frac{d-2}{2}}}{h^{\frac{d-2}{2}}}\sum_{i=1}^{n} \tilde{l}_i\mu_i^2 d\phi_i\right). \quad \text{(C.2)}$$

The above substitution rules should enable the reader to easily write down the general euclidean case, and we confine ourselves with the explicit form for the case of one angular momentum $\tilde{l} \equiv \tilde{l}_1 \neq 0$ only

$$\begin{aligned}
ds^2 &= H^{-\frac{d-2}{D-2}}\left(f d\tau^2 + \sum_{i=1}^{p}(dy^i)^2\right) + H^{\frac{p+1}{D-2}}\left(\bar{f}^{-1}\frac{r^2-\tilde{l}^2\cos^2\theta}{r^2-\tilde{l}^2}dr^2 + \Lambda_{\alpha\beta}d\eta^\alpha d\eta^\beta\right) - \\
&\quad - 2H^{-\frac{d-2}{D-2}}\frac{1}{1-\frac{\tilde{l}^2\cos^2\theta}{r^2}}\frac{h^{\frac{d-2}{2}}r_0^{\frac{d-2}{2}}}{r^{d-2}}\tilde{l}\sin^2\theta d\tau d\phi_1\,, \qquad \text{(C.3a)}\\
e^\phi &= H^{a/2}\,, \qquad \text{(C.3b)}\\
A_{p+1} &= -i(-1)^p\left(H^{-1}d\tau - \frac{r_0^{\frac{d-2}{2}}}{h^{\frac{d-2}{2}}}\tilde{l}\sin^2\theta d\phi_1\right)\wedge dy^1\wedge dy^2\wedge\cdots\wedge dy^p\,, \qquad \text{(C.3c)}
\end{aligned}$$

where

$$H = \frac{1}{1-\frac{\tilde{l}^2\cos^2\theta}{r^2}}\frac{h^{d-2}}{r^{d-2}}\,, \qquad f = 1 - \frac{1}{1-\frac{\tilde{l}^2\cos^2\theta}{r^2}}\frac{r_0^{d-2}}{r^{d-2}}\,, \qquad \bar{f} = 1 - \frac{1}{1-\frac{\tilde{l}^2}{r^2}}\frac{r_0^{d-2}}{r^{d-2}} \qquad \text{(C.4)}$$

and the expressions for $\Lambda_{\alpha\beta}$ in the one-angular momentum case can be found in appendix B.

D. Change of variables from $(r_0, \{l_i\})$ to $(T, \{\Omega_i\})$

In this appendix we give the formulae needed to go from the supergravity variables $(r_0, \{l_i\})$ to the thermodynamic quantities $(T, \{\Omega_i\})$. Since it is not possible to obtain closed expressions (for general d) for this change of variables, we perform this analysis in a weak angular momentum expansion

$$\frac{l_i}{r_0} \ll 1 \qquad \text{(D.1)}$$

keeping the first three terms only, which suffices for the applications of the text.

We use expressions (3.4) for $(T, \{\Omega_i\})$,

$$T = \frac{d-2-2\kappa}{4\pi r_H}\frac{r_0^{\frac{d-2}{2}}}{h^{\frac{d-2}{2}}}\,, \qquad \Omega_i = \frac{l_i}{(l_i^2+r_H^2)}\frac{r_0^{\frac{d-2}{2}}}{h^{\frac{d-2}{2}}} \qquad \text{(D.2)}$$

to compute the quantities $(r_0, \{l_i\})$ in terms of the former. For this we first need to use the relation (2.8) determining the horizon radius r_H in terms of these, and we find

$$\begin{aligned}
r_H = r_0\Bigg[1 &- \frac{1}{d-2}\sum_i\left(\frac{l_i}{r_0}\right)^2 - \frac{3}{2(d-2)^2}\left(\sum_i\left(\frac{l_i}{r_0}\right)^2\right)^2 + \\
&+ \frac{1}{2(d-2)}\sum_i\left(\frac{l_i}{r_0}\right)^4 + \cdots\Bigg]\,. \qquad \text{(D.3)}
\end{aligned}$$

Substituting this in (D.2) we obtain the expressions

$$T = \frac{d-2}{4\pi}\frac{r_0^{(d-4)/2}}{h^{(d-2)/2}}\left[1 - \frac{1}{d-2}\sum_i\left(\frac{l_i}{r_0}\right)^2 - \frac{7}{2(d-2)^2}\left(\sum_i\left(\frac{l_i}{r_0}\right)^2\right)^2 + \right. \\ \left. + \frac{3}{2(d-2)}\sum_i\left(\frac{l_i}{r_0}\right)^4 + \cdots\right], \tag{D.4a}$$

$$\Omega_i^2 = \frac{r_0^{d-4}}{h^{d-2}}\left(\frac{l_i}{r_0}\right)^2\left[1 + \frac{4}{d-2}\sum_j\left(\frac{l_j}{r_0}\right)^2 - 2\left(\frac{l_i}{r_0}\right)^2 + \cdots\right], \tag{D.4b}$$

which can be inverted to give

$$r_0 = \left(\tilde{T}h^{\frac{(d-2)}{2}}\right)^{\frac{2}{(d-4)}}\left[1 + \frac{2}{(d-4)(d-2)}\sum_i\tilde{\omega}_i^2 - \frac{2(2d-9)}{(d-2)^2(d-4)^2}\left(\sum_i\tilde{\omega}_i^2\right)^2 + \right. \\ \left. + \frac{1}{(d-2)(d-4)}\sum_i\tilde{\omega}_i^4 + \cdots\right], \tag{D.5a}$$

$$\left(\frac{l_i}{r_0}\right)^2 = \tilde{\omega}_i^2\left[1 - \frac{6}{d-2}\sum_j\tilde{\omega}_j^2 + 2\tilde{\omega}_i^2 + \cdots\right], \tag{D.5b}$$

where we have defined

$$\tilde{T} \equiv \frac{4\pi T}{d-2}, \qquad \tilde{\omega}_i = \frac{\Omega_i}{\tilde{T}} \tag{D.6}$$

Finally, we also give the expression

$$r_0^{d-2} = \left(\tilde{T}h^{(d-2)/2}\right)^{2(d-2)/(d-4)}\left[1 + \frac{2}{d-4}\sum_i\tilde{\omega}_i^2 - \right. \\ \left. - \frac{2(d-6)}{(d-2)(d-4)^2}\left(\sum_i\tilde{\omega}_i^2\right)^2 + \frac{1}{d-4}\sum_i\tilde{\omega}_i^4 + \cdots\right] \tag{D.7}$$

which enters the free energy (3.11).

E. Polylogarithms

In this appendix we define the polylogarithm functions and give some general properties. We also discuss a continuation of the polylogarithms to real numbers greater than one, which is used in section 5.1. The nth polylogarithm function is defined as

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n}, \tag{E.1}$$

for $z \in \mathbb{C} - \{u \in \mathbb{R} + 2\pi i\mathbb{Z} | \operatorname{Re}(u) > 1\}$, where $\operatorname{Re}(u)$ means the real part of u. satisfying

$$\mathrm{Li}_n(1) = \zeta(n) \text{ for } n \neq 1\,. \tag{E.2}$$

We also have the relation

$$\mathrm{Li}_n(-1) = \tilde{\zeta}(n)\,, \tag{E.3}$$

where we have defined

$$\tilde{\zeta}(n) = \begin{cases} (1-2^{1-n})\zeta(n)\,, & n \neq 1\,, \\ -\log(2)\,, & n = 1\,. \end{cases} \tag{E.4}$$

The polylogarithm satisfies the integral formula

$$\int_0^\infty dx\, x^{n-2} \log(1-e^{z-x}) = -\Gamma(n-1)\mathrm{Li}_n(e^z)\,, \tag{E.5}$$

for $z \in \{u \in \mathbb{C} | \operatorname{Im}(u) \notin 2\pi\mathbb{Z}\}$.

We also define

$$B_n(z) = \frac{1}{2}\left(\mathrm{Li}_n(e^z) + \mathrm{Li}_n(e^{-z})\right), \tag{E.6}$$

for $z \in \{u \in \mathbb{C} | \operatorname{Im}(u) \in [-\pi, \pi] - \{0\}\}$, and

$$F_n(z) = \frac{1}{2}\left(\mathrm{Li}_n(-e^z) + \mathrm{Li}_n(-e^{-z})\right), \tag{E.7}$$

for $z \in \{u \in \mathbb{C} | \operatorname{Im}(u) \in (-\pi, \pi)\}$.

For even n we have

$$B_n(z) = \sum_{k=0}^{n/2} \zeta(n-2k)\frac{z^{2k}}{(2k)!} \pm \frac{i\pi}{2}\frac{z^{n-1}}{(n-1)!}\,, \tag{E.8}$$

where the optional sign is the sign of $\operatorname{Im}(z)$. For odd n we have

$$\begin{aligned} B_n(z) =& \sum_{k=0}^{\frac{n-3}{2}} \zeta(n-2k)\frac{z^{2k}}{(2k)!} + \left(\pm\frac{i\pi}{2} + \sum_{k=1}^{n-1}\frac{1}{k} - \log(z)\right)\frac{z^{n-1}}{(n-1)!} + \\ &+ \sum_{k=\frac{n+1}{2}}^{\infty} \zeta(n-2k)\frac{z^{2k}}{(2k)!}\,, \end{aligned} \tag{E.9}$$

where the optional sign is the sign of $\operatorname{Im}(z)$. These functions satisfy

$$B_{n-2}(z) = \frac{d^2}{dz^2}B_n(z)\,, \tag{E.10}$$

for any n. Considering F_n, we find that for even n we have

$$F_n(z) = \sum_{k=0}^{n/2} \tilde{\zeta}(n-2k)\frac{z^{2k}}{(2k)!} \tag{E.11}$$

and for odd n we have

$$F_n(z) = \sum_{k=0}^{\infty} \tilde{\zeta}(n-2k)\frac{z^{2k}}{(2k)!}\,. \tag{E.12}$$

It is easy to see that

$$F_{n-2}(z) = \frac{d^2}{dz^2}F_n(z)\,, \tag{E.13}$$

for any n.

If we want to define $B_n(z)$ also for $z \in \mathbb{R}$ we can note that while the imaginary part of $B_n(z)$ changes sign when crossing the real line, the real part is continuous. Thus, it is natural to define B_n on the real line as the limit of the real part of B_n. This is also what the principal value prescription gives, since this is $\frac{1}{2}(B(x+i\epsilon)+B(x-i\epsilon))$ for $\epsilon \to 0^+$ with $x \in \mathbb{R}$. Thus, let $x \in \mathbb{R}$, then for even n we write

$$B_n(x) = \sum_{k=0}^{n/2} \zeta(n-2k)\frac{x^{2k}}{(2k)!} \tag{E.14}$$

and for odd n we write

$$B_n(x) = \sum_{k=0}^{\frac{n-3}{2}} \zeta(n-2k)\frac{x^{2k}}{(2k)!} + \left(\sum_{k=1}^{n-1}\frac{1}{k} - \log(x)\right)\frac{x^{n-1}}{(n-1)!} + \sum_{k=\frac{n+1}{2}}^{\infty} \zeta(n-2k)\frac{x^{2k}}{(2k)!}\,. \tag{E.15}$$

Again, these functions satisfy

$$B_{n-2}(x) = \frac{d^2}{dx^2}B_n(x)\,, \tag{E.16}$$

for any n.

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